

Bounds on the effect of perturbations of continuum random Schrödinger operators and applications

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ABSTRACT. We consider an alloy-type random Schrödinger operator H in multi-dimensional Euclidean space \mathbb{R}^d and its perturbation H^τ by a bounded and compactly supported potential with coupling constant τ . Our main estimate concerns separate exponential decay of the disorder-averaged Schatten-von Neumann p -norm of $\chi_a(f(H) - f(H^\tau))\chi_b$ in a and b . Here, χ_a is the multiplication operator corresponding to the indicator function of a unit cube centred about $a \in \mathbb{R}^d$, and f is in a suitable class of functions of bounded variation with distributional derivative supported in the region of complete localisation for H . The estimates also hold uniformly for Dirichlet restrictions of the pair of operators to arbitrary open regions. Estimates of this and related type have applications towards the spectral shift function for the pair (H, H^τ) and towards Anderson orthogonality. For example, they imply that in the region of complete localisation the spectral shift function almost surely coincides with the index of the corresponding pair of spectral projections. Concerning the second application, we show that both presence and absence of Anderson orthogonality occur with positive probability for energies in the region of complete localisation. This leads to new criteria for detecting spectral phases of random Schrödinger operators different from the region of complete localisation.

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1. Introduction

One way to learn about a physical system is to study its reactions to external perturbations. The fluctuation-dissipation theorem of linear-response theory in statistical mechanics provides a classical example for this approach. Simon and Wolff [SW86] adhered to the same idea when they derived powerful localisation and delocalisation criteria in the theory of random Schrödinger operators [Kir89, CL90, PF92, Kir08, AW15]. In this paper, we analyse the response of continuum random Schrödinger operators to a perturbation in the region of complete localisation. The response can be used as a criterion for the existence of other spectral phases.

More precisely, we estimate the probabilistic expectation of the Schatten-von Neumann p -norm of $\chi_a(f(H_G) - f(H_G^\tau))\chi_b$. Here, H_G is the Dirichlet restriction of an alloy-type random Schrödinger operator in multi-dimensional Euclidean space \mathbb{R}^d to an open region $G \subseteq \mathbb{R}^d$. The operator H_G^τ is a perturbation of H_G by a bounded and compactly supported potential with coupling constant τ , χ_a is the multiplication operator corresponding to the indicator function of a unit cube centred about $a \in \mathbb{R}^d$ and f is a function of bounded variation on \mathbb{R} whose support is bounded from the right and which is constant except for energies in a compact interval inside the region of complete localisation for H . The easiest non-trivial example in this class of functions is the indicator function $f = \mathbb{1}_{(-\infty, E]}$, where E is in the region of complete localisation. In Theorem 3.1 we find the bound

$$\mathbb{E}[\|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|_p] \leq C|\tau|^{s/2} \text{TV}(f) e^{-\mu(|a|+|b|)} \quad (1.1)$$

with constants $C, \mu > 0$ that depend on $s \in (0, 1)$ and $p \in (0, \infty)$, but are uniform in the open region $G \subseteq \mathbb{R}^d$. Since the perturbation is compactly supported in space, the bound exhibits a separate exponential decay in both a and b . It is proportional to the total-variation norm $\text{TV}(f)$ of f and vanishes algebraically in the limit of small coupling constants. This local estimate is turned into the global estimate

$$(\mathbb{E}[\|f(H_G) - f(H_G^\tau)\|_p^q])^{1/q} \leq C_\tau \text{TV}(f) \quad (1.2)$$

in Corollary 3.4. In principle, this is done by summing over a and b . Here, $q \in (0, \infty)$ is arbitrary, and the constant C_τ again vanishes in the limit as $\tau \downarrow 0$. Analogous estimates hold for perturbations by boundary conditions and are stated in Theorems 3.2 and 3.6. In particular, they allow to control macroscopic limits of differences $f(H_L) - f(H_L^\tau)$ in Schatten-von Neumann p -norms for finite-volume operators restricted to boxes of size L .

There is a vast literature on Schatten-von Neumann-class properties of operator differences $f(H) - f(H^\tau)$, where H and H^τ satisfy assumptions which are often

motivated by scattering theory. For example, if the unperturbed operator is given by the Laplacian $-\Delta$ and the potential V decays sufficiently fast at infinity, the function f needs to have certain differentiability properties in order to ensure that $f(-\Delta) - f(-\Delta + V)$ is trace class; see e.g. [Yaf92, FP15] and references therein. We emphasise that we are concerned with the opposite regime in this paper. In the region of complete localisation for random Schrödinger operators, even discontinuous functions f lead to super-trace-class properties of the operator difference $f(H) - f(H^\tau)$.

Our proofs of (1.1) and the related results in Sect. 3.1 are structural in that they do not depend on the details of the model. They exploit general properties of Schrödinger operators such as the deterministic a priori bounds (B.6) for bounded functions of $H^{(\tau)}$ and the Combes-Thomas estimate. In addition, localisation enters through exponentially decaying fractional-moment bounds (2.6) for the resolvents.

Before we turn to applications, we mention two auxiliary results in our proofs whose validity is not restricted to the region of complete localisation and which may be of independent interest in different contexts. Firstly, in order to express $f(K)$, K a self-adjoint operator, in terms of the resolvent we formulate a suitably adapted version of the Helffer-Sjöstrand formula in Lemma 4.1 which applies to compactly supported functions of bounded variation. In order to treat such general functions f we have traded regularity of f against boundedness of (spatially localised) resolvents. In the case $K = H^{(\tau)}$, the latter is provided by the a priori bounds (2.7). The second auxiliary result is a scale-free unique continuation principle for averaged local traces of non-negative functions of *infinite-volume* ergodic random Schrödinger operators in Lemma 5.3. It follows from a scale-free unique continuation principle for spectral projections of *finite-volume* random Schrödinger operators [Kle13, NTTV15] by a suitable limit procedure. We are not aware that this consequence has been noticed elsewhere.

We apply the estimates (1.1), (1.2) and their relatives in two different ways. Firstly, a particular choice for the function f is the indicator function $f = \mathbb{1}_{(-\infty, E]}$ with E lying inside the region of complete localisation. In this way we derive properties of the spectral shift function for random Schrödinger operators in the region of complete localisation. In Theorem 3.10 we establish convergence of the finite-volume spectral shift functions towards the infinite-volume spectral shift function and identify the latter with the index of the corresponding pair of spectral projections. We also show Hölder continuity of the averaged spectral shift function and find a *lower* bound in Theorem 3.16 which establishes strict positivity under appropriate conditions. This bound relies on the latest quantitative unique continuation principle for spectral projections of random Schrödinger operators [Kle13, NTTV15].

The second application relates to the phenomenon known as Anderson orthogonality in the physics literature. Anderson orthogonality is a fundamental property of fermionic systems [And67, OT90, Mah00]. Given two non-interacting systems of N Fermions, which differ by a local perturbation, it is defined as the vanishing of the overlap

$$S_{N,L} := |\langle \Phi_{N,L}, \Psi_{N,L} \rangle_{\mathcal{H}_{N,L}}| \quad (1.3)$$

of their N -particle ground states $\Phi_{N,L}$ and $\Psi_{N,L}$ in the thermodynamic limit $L \rightarrow \infty$, $N \rightarrow \infty$ with a non-vanishing particle density $N/L^d \rightarrow \rho(E) > 0$, which we parametrise in terms of the Fermi energy E . In the above, $\langle \cdot, \cdot \rangle_{\mathcal{H}_{N,L}}$

denotes the scalar product on the fermionic N -particle Hilbert space. Despite the attention Anderson orthogonality had and still has within the physics community, first mathematical results are only rather recent. There are by now several works [KOS14, GKM14, GKMO16] which prove vanishing upper bounds on $S_{N,L}$ and, in more special one-dimensional situations, even exact asymptotics [KOS15, Geb15]. The common conceptional point of these works is that if the Fermi energy belongs to the absolutely continuous spectrum of the operator H , then there is algebraic decay of $S_{N,L}$ in the thermodynamic limit. In contrast, no attention has been paid to Anderson orthogonality for random Schrödinger operators with Fermi energy in the region of complete localisation, as it was established wisdom that the effect would not occur in this case [GBLA02]. It was only the very recent studies [KNS15, DPLDS15] which revealed interesting phenomena arising in this situation: (i) Anderson orthogonality does occur for random Schrödinger operators when the Fermi energy is in the region of complete localisation, but with a probability strictly between 0 and 1. (ii) If $S_{N,L}$ vanishes in the thermodynamic limit, it vanishes even exponentially fast in the system size. In this paper, we provide a mathematical understanding of most of these findings. For a Fermi energy E in the region of complete localisation, we prove in Theorem 3.19 that the finite-volume overlap $S_{N,L}$ converges in expectation and almost surely to a random limit $S(E)$, given by a Fredholm determinant related to the spectral shift operator $\mathbb{1}_{(-\infty, E]}(H) - \mathbb{1}_{(-\infty, E]}(H^\tau)$. Anderson orthogonality occurs if and only if the spectral shift operator has an eigenvalue of modulus 1. If $S(E) \neq 0$, we prove exponential speed of convergence in the thermodynamic limit. We also identify conditions where both presence and absence of Anderson orthogonality occur with positive probability.

Finally, the results of this paper lead to new criteria which offer a possibility to detect spectral phases of random Schrödinger operators other than the region of complete localisation. The first one is about the divergence of the averaged Schatten- p -norm of finite-volume spectral shift operators in the macroscopic limit. It is formulated in Corollary 3.8. The second criterion, see Corollary 3.21, concerns the almost-sure occurrence of Anderson orthogonality along a sequence of sufficiently fast growing length scales, no matter how small the coupling constant of the perturbation potential is.

The plan of this paper is as follows. In the next section we introduce the alloy-type Anderson model and fix our notation. Section 3 contains the results. In Section 3.1 we present bounds on disorder-averaged Schatten-von Neumann norms for differences of functions of the unperturbed and perturbed random Schrödinger operator, which hold in the region of complete localisation. We proceed by describing two applications of these bounds: in Section 3.2 we are concerned with applications towards the spectral shift function. In Section 3.3 we study the implications for Anderson orthogonality in the region of complete localisation. All proofs are deferred to Sections 4 to 6. In Appendix A we verify the stability of fractional moment bounds under local perturbations. Schatten von Neumann-class estimates for Schrödinger operators are reviewed in Appendix B. Finally, we collect some properties of the index of a pair of spectral projections in Appendix C.

2. The model

We consider a random Schrödinger operator

$$\omega \mapsto H_\omega := H_0 + V_\omega := H_0 + \lambda \sum_{k \in \mathbb{Z}^d} \omega_k u_k \quad (2.1)$$

acting on a dense domain in the Hilbert space $L^2(\mathbb{R}^d)$ for $d \in \mathbb{N}$ and $\lambda \geq 0$. Here, H_0 is a non-random self-adjoint operator and $\omega \mapsto V_\omega$ is a random alloy-type potential subject to the following assumptions.

- (K) The non-random operator $H_0 := -\Delta + V_0$ consists of the non-negative Laplacian $-\Delta$ on d -dimensional Euclidean space \mathbb{R}^d and of a deterministic, \mathbb{Z}^d -periodic and bounded background potential $V_0 \in L^\infty(\mathbb{R}^d)$.
- (V1) The family of canonically realised random coupling constants $\omega := (\omega_k)_{k \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ is distributed independently and identically according to the Borel probability measure $\mathbb{P} := \bigotimes_{\mathbb{Z}^d} P_0$ on $\mathbb{R}^{\mathbb{Z}^d}$. We write \mathbb{E} for the corresponding expectation. The single-site distribution P_0 is absolutely continuous with respect to Lebesgue measure on \mathbb{R} and the corresponding Lebesgue density ρ obeys $0 \leq \rho \in L_c^\infty(\mathbb{R})$ and $\text{supp}(\rho) \subseteq [0, 1]$.
- (V2) The single-site potentials $u_k(\cdot) := u(\cdot - k)$, $k \in \mathbb{Z}^d$, are translates of a non-negative function $0 \leq u \in L_c^\infty(\mathbb{R}^d)$. Moreover, there exists a constant $C_{u,-}$ such that the covering condition

$$0 < C_{u,-} \leq \sum_{k \in \mathbb{Z}^d} u_k \quad (2.2)$$

holds.

The assumption $\text{supp}(\rho) \subseteq [0, 1]$ on the support of the single-site probability density is made for convenience only and is not stronger than requiring $\text{supp}(\rho)$ to be compact. The regularity of P_0 as well as the covering condition (2.2) guarantee that the fractional-moment bounds (2.6) and the a priori bounds (2.7) below hold in fair generality for our model, see also Remarks 2.2 below.

Throughout the paper we drop the subscript ω from H and other quantities when we think of these quantities as random variables (as opposed to their particular realisations). We write *almost surely* if an event occurs \mathbb{P} -almost surely, and, given an E -dependent statement for $E \in B \in \text{Borel}(\mathbb{R})$, we write *for a.e. $E \in B$* , if this statement holds for Lebesgue-almost every $E \in B$.

The alloy-type random Schrödinger operator (2.1) is \mathbb{Z}^d -ergodic with respect to lattice translations. It follows that there exists a closed set $\Sigma \subset \mathbb{R}$, the non-random spectrum of H , such that $\Sigma = \sigma(H)$ holds almost surely [PF92]. Here, $\sigma(H)$ is the spectrum of H . Such a statement also holds individually for the components of the spectrum in the Lebesgue decomposition.

Given an open subset $G \subseteq \mathbb{R}^d$, we write H_G for the Dirichlet restriction of H to G . Whenever it is convenient, we think of a finite-volume operator H_G to act in $L^2(\mathbb{R}^d)$ by being trivially extended on $L^2(\mathbb{R}^d \setminus G)$. We define the random finite-volume eigenvalue counting function

$$\mathbb{R} \ni E \mapsto N_L(E) := \text{tr}(\mathbf{1}_{(-\infty, E]}(H_L)) \quad (2.3)$$

for $L > 0$, where $\mathbb{1}_B$ stands for the indicator function of a set $B \in \text{Borel}(\mathbb{R})$, $H_L := H_{\Lambda_L}$ and $\Lambda_L := (-L/2, L/2)^d$ for the open cube about the origin of side-length L . We write $|\Lambda_L| = L^d$ for the Lebesgue volume of the latter. The regularity assumption (V1) on the single-site distribution ensures a (standard) Wegner estimate [CHK07a]. The Wegner estimate and ergodicity imply that the limit

$$\mathcal{N}(E) := \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} N_L(E) \quad (2.4)$$

exists for all $E \in \mathbb{R}$ almost surely. The limit function \mathcal{N} is called the *integrated density of states* of H . Moreover, the Wegner estimate implies that it is an absolutely continuous function under our assumptions. Its Lebesgue derivative \mathcal{N}' is called the *density of states*. We refer to, e.g., [KM07, Ves08] for an overview.

This paper is concerned with local perturbations of the random Schrödinger operator H . For a bounded and compactly supported function $W \in L_c^\infty(\mathbb{R}^d)$, not necessarily of definite sign, we set

$$H^\tau := H + \tau W, \quad (2.5)$$

where $\tau \in [0, 1]$ is the tunable strength of the perturbation and $H^0 = H$ is the unperturbed operator. We assume, from now on, that W is fixed. The dependence of our results on the strength of the perturbation will be analysed through the dependence on τ . We note that H^τ has the same integrated density of states as H .

All results in this paper refer to energies in the *region of complete localisation* – following the terminology of [GK06]. In the continuum setting, it has mostly been studied by the multi-scale analysis. We refer to [GK01, BK05, GHK07, GK13] for newer developments in this direction and to the literature cited therein. In this paper, we characterise the region of complete localisation in terms of fractional-moment bounds [AENSS06], see also [BNSS06]. To this end we need to introduce some more notation. Given a self-adjoint operator A , we denote by $R_z(A) := (A - z)^{-1}$ its resolvent at $z \in \mathbb{C} \setminus \sigma(A)$. We write $Q_a := a + [-1/2, 1/2]^d$ for the (closed) unit cube centred at $a \in \mathbb{R}^d$ and $\chi_a := \mathbb{1}_{Q_a}$ for its indicator function. We denote the supremum norm on \mathbb{R}^d by $|x| := \sup_{j=1, \dots, d} |x_j|$, where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Finally, $\text{dist}(U, V) := \inf\{|x - y| : x \in U, y \in V\}$ stands for the distance of two subsets $U, V \subset \mathbb{R}^d$ with respect to the supremum norm.

Definition 2.1 (Fractional-moment bounds). We write $E \in \Sigma_{\text{FMB}} := \Sigma_{\text{FMB}}(H)$ if there exists a neighbourhood $U := U_E$ of E in \mathbb{R} and an exponent $0 < s < 1$ such that the following holds: there exist constants $C, \mu > 0$ such that for all open subsets $G \subseteq \mathbb{R}^d$ and all $a, b \in \mathbb{R}^d$ the bound

$$\sup_{\substack{E' \in U_E \\ \eta \neq 0}} \mathbb{E} [\|\chi_a R_{E' + i\eta}(H_G) \chi_b\|^s] \leq C e^{-\mu|a-b|} \quad (2.6)$$

holds true. Here, $\|\cdot\|$ stands for the operator norm.

Remarks 2.2. (i) Suppose that $I \subset \Sigma_{\text{FMB}}$ for a compact set $I \subset \mathbb{R}$. Then there exist constants C, μ such that (2.6) holds uniformly for every $E \in I$ with these constants.

(ii) Suppose that (2.6) holds for *some* exponent $0 < s < 1$, then it holds for *all* exponents $0 < s < 1$ with constants C and μ depending on s . The original proof

of this fact in [ASFH01, Lemma B.2] for discrete models extends to the continuum setting, see Lemma A.2.

(iii) We show in Lemma A.1 that the stability of the regime of complete localisation $\Sigma_{\text{FMB}}(H) = \Sigma_{\text{FMB}}(H^\tau)$ holds for perturbations τW as considered in this paper. Hence, even though our proofs require mostly fractional moment bounds for both H and H^τ , it suffices to postulate (2.6) for the unperturbed operator H only and – according to the previous remark – with a fixed exponent s .

(iv) Bounds of the form (2.6) have first been derived for the lattice Anderson model in [AM93], see also [AG98, ASFH01], either for sufficiently strong disorder or in the Lifshitz-tail regime. They were generalized to continuum random Schrödinger operators in [AENSS06]. The formulation there differs with respect to the distance function that is used in (2.6). We refer to [AENSS06, (8) in App. A] for an interpretation. Bounds as in (2.6) have been derived in [BNSS06] for the fluctuation-boundary regime by adapting the methods from [AENSS06]. Fractional-moment bounds for the resolvent imply Anderson localisation in the strongest form, see e.g. [GK06, AW15].

(v) The fractional-moment bounds (2.6) hold for energies below $\inf(\Sigma)$ as follows from a Combes-Thomas estimate [GK03, Cor. 1], which extends to finite-volume operators.

(vi) The proof of (2.6) in [AENSS06] for alloy-type random Schrödinger operators relies – as in the discrete case – on a priori estimates for fractional moments of the resolvent. In fact, the following holds for our model: for every $0 < s < 1$ and every bounded interval $I \subset \mathbb{R}$ there exists a finite constant $C_s > 0$ such that

$$\sup_{\tau \in [0,1]} \sup_{\substack{E \in I, \eta \neq 0 \\ G \subseteq \mathbb{R}^d, a, b \in \mathbb{R}^d}} \mathbb{E} [\|\chi_a R_{E+i\eta}(H_G^\tau) \chi_b\|^s] \leq C_s. \quad (2.7)$$

Except for the supremum in τ , this is the content of Lemma 3.3 and the subsequent Remark (4) in [AENSS06]. The relevant τ -dependence of the constant in that lemma enters in (3.31) upon replacing z by $z - \tau W$. This leads to a polynomial τ -dependence of the following norm estimates and implies uniformity in $\tau \in [0, 1]$.

3. Results

The results in Subsection 3.1 are at the heart of the proofs for the applications on the spectral shift function and Anderson's orthogonality in the following two subsections.

3.1. Schatten-von Neumann-class estimates. First, we fix our notation. Given a compact operator A on a separable Hilbert space, we order its singular values $b_1(A) \geq b_2(A) \geq \dots$ in a non-increasing way. For $p > 0$ we say that $A \in \mathcal{S}^p$, the *Schatten-von Neumann p -class*, if

$$\|A\|_p := (\text{tr}(|A|^p))^{1/p} = \left(\sum_{n \in \mathbb{N}} b_n(A)^p \right)^{1/p} < \infty. \quad (3.1)$$

Definition (3.1) induces a complete norm on the linear space \mathcal{S}^p for $p \geq 1$, while for $0 < p < 1$ it induces a complete quasi-norm for which the adapted triangle inequality

$$\|A + B\|_p^p \leq \|A\|_p^p + \|B\|_p^p \quad (3.2)$$

holds [McC67, Thm. 2.8]. We state our results for functions f on \mathbb{R} which belong to the space of functions of bounded variation

$$\text{BV}(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} \text{ measurable} : \text{TV}(f) < \infty\}. \quad (3.3)$$

Here, the *total variation* is given by $\text{TV}(f) := \sup_{(x_p)_p \in \mathcal{P}} \sum_p |f(x_{p+1}) - f(x_p)|$ with the supremum being taken over the set \mathcal{P} of all finite partitions of \mathbb{R} . We write $\text{BV}_c(\mathbb{R})$ for the subspace of all functions in $\text{BV}(\mathbb{R})$ with compact support. Finally, given a bounded interval $I = [I_-, I_+] \subset \mathbb{R}$, we define the vector space of functions

$$\mathcal{F}_I := \left\{ f \in \text{BV}(\mathbb{R}) : f|_{(-\infty, I_-]} \equiv \text{const.}, f|_{[I_+, \infty)} \equiv 0 \right\} \subset L^\infty(\mathbb{R}), \quad (3.4)$$

which is a normed space with respect to the total variation.

Theorem 3.1. *Fix $p > 0$, $0 < s < 1$ and let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then there exist finite constants $C, \mu > 0$ such that the following holds: for all $G \subseteq \mathbb{R}^d$ open, $a, b \in \mathbb{R}^d$, $\tau \in [0, 1]$ and $f \in \mathcal{F}_I$ we have*

$$\mathbb{E}[\|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|_p] \leq C|\tau|^{s/2} \text{TV}(f) e^{-\mu(|a|+|b|)}. \quad (3.5)$$

The method we use in the proof of Theorem 3.1 can also be applied to treat a perturbation by a boundary condition.

Theorem 3.2. *Fix $p > 0$ and let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then there exist finite constants $C, \mu > 0$ such that the following holds: for all open subsets $G \subset \tilde{G} \subseteq \mathbb{R}^d$ with $\text{dist}(\partial\tilde{G}, \partial G) \geq 1$, for all $a, b \in \mathbb{R}^d$ such that $Q_a \cap G \neq \emptyset$ or $Q_b \cap G \neq \emptyset$, for all $\tau \in [0, 1]$ and all $f \in \mathcal{F}_I$ we have*

$$\mathbb{E}[\|\chi_a(f(H_G^\tau) - f(H_{\tilde{G}}^\tau))\chi_b\|_p] \leq C \text{TV}(f) e^{-\mu[\text{dist}(a, \partial G) + \text{dist}(b, \partial G)]}. \quad (3.6)$$

Remarks 3.3. (i) The separate exponential decay of (3.5) in a and b reflects that the operator $f(H_G) - f(H_G^\tau)$ is exponentially small away from the support of $H_G - H_G^\tau$. An analogous comment applies to (3.6).

(ii) Choosing $\tilde{G} = \mathbb{R}^d$, Theorem 3.2 controls the macroscopic limit of $f(H_G^\tau)$.

(iii) From a structural point of view it is only the fractional-moment bound (2.6) and the a priori bound (2.7) rather than the precise form of the Schrödinger operator which enter in our proofs of Theorems 3.1 and 3.2. Similar methods are used in [DGHKM] to treat a change of boundary conditions from Dirichlet to Neumann.

(iv) Our proof of the above results relies on an application of the Helffer-Sjöstrand formula, Corollary 4.3 below, to express the operators $f(H_G)$ and $f(H_G^\tau)$ in terms of the resolvents of H_G and H_G^τ . To handle functions $f \in \mathcal{F}_I$, which may have jump discontinuities, we elaborate on the assumptions of the Helffer-Sjöstrand formula by trading regularity of f against regularity of the resolvent, see Lemma 4.1. In the case of random Schrödinger operators this regularity is provided by the a priori bound (2.7).

A consequence of Theorem 3.1 is

Corollary 3.4. *Fix $p, q > 0$ and $\tau \in [0, 1]$. Let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then there exists a finite constant $C_\tau > 0$ such that for all $G \subseteq \mathbb{R}^d$ open and $f \in \mathcal{F}_I$*

$$(\mathbb{E}[\|f(H_G) - f(H_G^\tau)\|_p^q])^{1/q} \leq C_\tau \text{TV}(f). \quad (3.7)$$

The constant C_τ vanishes algebraically in τ as $\tau \downarrow 0$.

Remark 3.5. If U is a bounded and sufficiently fast decaying potential, it is shown in [FP15] that the operator difference $f(-\Delta) - f(-\Delta + U) \in \mathcal{S}^1$ for functions f in an appropriate Besov space of weakly differentiable functions. In contrast, for $f = \mathbb{1}_{(-\infty, E]}$, $E \in \mathbb{R}$, the operator $\mathbb{1}_{(-\infty, E]}(-\Delta) - \mathbb{1}_{(-\infty, E]}(-\Delta + U)$ exhibits bands of absolutely continuous spectrum whose widths is determined by the scattering phase shifts at energy E [Pus08]. In contrast, Corollary 3.4 shows that if $-\Delta$ is replaced by a random Schrödinger operator, $E \in \Sigma_{\text{FMB}}$ and U has compact support, then $\mathbb{1}_{(-\infty, E]}(H) - \mathbb{1}_{(-\infty, E]}(H^\tau)$ belongs to any Schatten-von Neumann class. The next subsection deals with more applications towards the spectral shift.

Theorem 3.1 and Theorem 3.2 together imply

Theorem 3.6. Fix $p, q > 0$ and let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then there exist finite constants $C, \mu > 0$ such that for all $f \in \mathcal{F}_I$, $\tau \in [0, 1]$ and $L > 0$

$$\left(\mathbb{E} \left[\left\| (f(H_L) - f(H_L^\tau)) - (f(H) - f(H^\tau)) \right\|_p^q \right] \right)^{1/q} \leq C \text{TV}(f) e^{-\mu L}. \quad (3.8)$$

Moreover, given a sequence $(L_n)_{n \in \mathbb{N}} \subset (0, \infty)$ of lengths with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$, then for every $\gamma \in (0, \mu)$ the convergence

$$\lim_{n \rightarrow \infty} e^{\gamma L_n} \left\| (f(H_{L_n}) - f(H_{L_n}^\tau)) - (f(H) - f(H^\tau)) \right\|_p = 0 \quad (3.9)$$

holds almost surely.

Remark 3.7. If $(X_n)_{n \in \mathbb{N}} \subset (0, \infty)$ is a sequence of random variables with summable expectation, $(\mathbb{E}[X_n])_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$, then $\lim_{n \rightarrow \infty} X_n = 0$ holds almost surely. This elementary fact shows that (3.9) follows directly from (3.8).

Corollary 3.4 or Theorem 3.6 offer a way to detect spectral phases of random Schrödinger operators other than the region of complete localisation.

Corollary 3.8. Let $E \in \Sigma$ and $p > 0$, then

$$\sup_{L > 0} \mathbb{E} [\|T(E, H_L, H_L^\tau)\|_p] = \infty \implies E \notin \Sigma_{\text{FMB}}. \quad (3.10)$$

Remark 3.9. The criterion of Corollary 3.8 should be compared to the behaviour of the spectral shift operator in the (hypothetical) conducting phase, provided $W \geq 0$: the finite-volume spectral shift operator $T(E, H_L, H_L^\tau)$ is finite rank for all $E \in \mathbb{R}$, hence even super trace class. Lemma C.1 then implies the representation

$$\begin{aligned} \|T(E, H_L, H_L^\tau)\|_{2n}^{2n} &= \|\mathbb{1}_{(-\infty, E]}(H_L) \mathbb{1}_{(E, \infty)}(H_L^\tau) \mathbb{1}_{(-\infty, E]}(H_L)\|_n^n \\ &\quad + \|\mathbb{1}_{(E, \infty)}(H_L) \mathbb{1}_{(-\infty, E]}(H_L^\tau) \mathbb{1}_{(E, \infty)}(H_L)\|_n^n \end{aligned} \quad (3.11)$$

for every $n \in \mathbb{N}$. Now, suppose we knew there exists a spectral interval $J \subset \mathbb{R}$ such that H has absolutely continuous spectrum in J almost surely. It follows from (3.11) and [GKMO16, Thm. 3.4] that, almost surely given any $p > 0$, $\|T(E, H_L, H_L^\tau)\|_p$ diverges at least logarithmically in L for a.e. $E \in J$ at which there is non-trivial scattering, i.e. where the quantity in (3.55) below is strictly positive. This demonstrates that the criterion (3.10) is potentially useful because its assumption holds in this case. The suppression of the logarithmic divergence in the region of complete localisation resembles the suppression of the logarithmic correction of the area law for the entanglement entropy observed in the discrete Anderson model in the region of complete localisation [PS14, EPS16].

3.2. Application 1: The spectral shift function. In this subsection we apply the findings of the previous subsection to prove convergence of the finite-volume spectral shift functions in the macroscopic limit for energies in the region of complete localisation. Moreover, we show that the limit, which is the infinite-volume spectral shift function, coincides with the trace of the spectral shift operator and also with the index of the corresponding pair of Fermi projections. In addition, we establish Hölder continuity and positivity of the disorder-averaged spectral shift function in the region of complete localisation.

First, we introduce the relevant quantities. Let A and B be self-adjoint operators in a Hilbert space. For any given $E \in \mathbb{R}$, we call

$$T(E, A, B) := \mathbb{1}_{(-\infty, E]}(A) - \mathbb{1}_{(-\infty, E]}(B) \quad (3.12)$$

a *spectral shift operator* (for A and B). Being a difference of two projections, we recall that

$$\|T(E, A, B)\| \leq 1. \quad (3.13)$$

Now, assume further that A and B are bounded from below and that $e^{-A} - e^{-B} \in \mathcal{S}^1$. Then there exists a unique function $\xi := \xi(\cdot, A, B) \in L^1_{\text{loc}}(\mathbb{R})$, the *spectral shift function* (for A and B), such that the equality

$$\text{tr}(f(A) - f(B)) = - \int_{\mathbb{R}} d\lambda f'(\lambda) \xi(\lambda, A, B) \quad (3.14)$$

holds for all test functions $f \in C^\infty(\mathbb{R})$ with $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$ and $\text{supp}(f')$ compact. We refer to, e.g., [Yaf92] for details and further properties. Finally, given two self-adjoint projections P and Q such that $\pm 1 \notin \sigma_{\text{ess}}(P - Q)$, the essential spectrum of $P - Q$, we define their *index* according to [ASS94, Prop. 3.1] as $\text{index}(P, Q) := \dim \ker(P - Q - 1) - \dim \ker(P - Q + 1)$. Further background on the index can be found in [ASS94]. In the particular case of spectral projections $P = \mathbb{1}_{(-\infty, E]}(A)$ and $Q = \mathbb{1}_{(-\infty, E]}(B)$, we write

$$\begin{aligned} \theta(E, A, B) &:= \text{index} \{ \mathbb{1}_{(-\infty, E]}(A), \mathbb{1}_{(-\infty, E]}(B) \} \\ &= \dim \ker(T(E, A, B) - \mathbb{1}) - \dim \ker(T(E, A, B) + \mathbb{1}). \end{aligned} \quad (3.15)$$

Next, we turn to the cases of interest for us, namely $A = H_{(L)}$ and $B = H_{(L)}^\tau$ for $L > 0$. We recall from Corollary 3.4 that

$$T(E, H_{(L)}, H_{(L)}^\tau) \in \mathcal{S}^1 \quad \text{almost surely for every } E \in \Sigma_{\text{FMB}} \quad (3.16)$$

and $L > 0$. Hence, the index $\theta(E, H_{(L)}, H_{(L)}^\tau)$ is well defined almost surely for every $E \in \Sigma_{\text{FMB}}$ and every $L > 0$. On the other hand, we infer from [HKNSV06, Thm. 1] that $e^{-H_{(L)}} - e^{-H_{(L)}^\tau} \in \mathcal{S}^1$ holds almost surely for all $L > 0$. The spectral shift function $\xi(\cdot, H_{(L)}, H_{(L)}^\tau)$ is therefore well-defined almost surely as a function in $L^1_{\text{loc}}(\mathbb{R})$.

For energies outside the essential spectrum of a pair of operators, it is known that all three quantities, the spectral shift function, the index and the trace of the spectral shift operator, are well-defined and coincide, see e.g. Prop. 2.1 and its proof in [Pus09]. We show in the next theorem that, in the case of the operators H and H^τ , the equality of the three quantities extends to the region of complete localisation.

Theorem 3.10. *Let $\tau \in [0, 1]$. Then,*

(i) *the spectral shift function, the trace of the shift operator and the Fredholm index coincide, i.e.*

$$\xi(E, H, H^\tau) = \text{tr} (T(E, H, H^\tau)) = \theta(E, H, H^\tau) \quad \text{for a.e. } E \in \Sigma_{\text{FMB}} \quad (3.17)$$

holds almost surely.

(ii) *given a compact interval $I \subset \Sigma_{\text{FMB}}$, there exist constants $C, \mu > 0$ such that the bound*

$$\mathbb{E} [|\xi(E, H_L, H_L^\tau) - \xi(E, H, H^\tau)|] \leq C e^{-\mu L} \quad \text{for a.e. } E \in I \quad (3.18)$$

holds for all $L > 0$. Moreover, given a sequence of lengths with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$, the convergence

$$\lim_{n \rightarrow \infty} \xi(E, H_{L_n}, H_{L_n}^\tau) = \xi(E, H, H^\tau) \quad \text{for a.e. } E \in I \quad (3.19)$$

holds almost surely.

Remarks 3.11. (i) It follows from [ASS94, Thm. 4.1] that (3.16) implies the equality

$$\text{tr} (T(E, H_{(L)}, H_{(L)}^\tau)) = \theta(E, H_{(L)}, H_{(L)}^\tau) \quad (3.20)$$

almost surely for every $E \in \Sigma_{\text{FMB}}$ and $L > 0$. The news in Theorem 3.10(i) is the left equality. The right equality demonstrates that all quantities take on an integer value.

(ii) In general, one cannot expect the spectral shift function and the index to coincide as soon as they are both well defined. For example, if E lies within the absolutely continuous spectrum of two operators, both the index and the spectral shift function may be well defined. But in this case, the spectral shift function cannot be expected to be integer-valued, whereas the index is by definition.

(iii) For the very same reason, the convergence (3.19) cannot hold in general. As there is no essential spectrum, the finite-volume spectral shift function is given as a difference of eigenvalue-counting functions

$$\xi(E, H_L, H_L^\tau) = \text{tr} (T(E, H_L, H_L^\tau)) = \text{tr} (\mathbb{1}_{(-\infty, E]}(H_L)) - \text{tr} (\mathbb{1}_{(-\infty, E]}(H_L^\tau)) \quad (3.21)$$

for all $E \in \mathbb{R}$ and are, thus, integer valued by definition. The infinite-volume quantity need not be. However, for a general Schrödinger operator $-\Delta + V_0$ and a perturbation by a bounded, compactly supported potential, vague convergence of the finite-volume spectral shift functions is known [HM10].

(iv) Theorem 3.10(ii) is formulated in terms of the spectral shift function. It holds for the index as well, even without having to exclude a Lebesgue null set of exceptional energies. Given a sequence of lengths with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$, the analogue of (3.19) reads

$$\lim_{n \rightarrow \infty} \theta(E, H_{L_n}, H_{L_n}^\tau) = \theta(E, H, H^\tau) \quad (3.22)$$

almost surely for every $E \in \Sigma_{\text{FMB}}$. This follows directly from (3.20) and an application of Theorem 3.6 to the function $f = \mathbb{1}_{(-\infty, E]}$.

The previous theorem immediately implies the boundedness of the finite-volume spectral shift functions with respect to the volume.

Corollary 3.12. *Let $\tau \in [0, 1]$. Given a sequence of lengths with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$\sup_{n \in \mathbb{N}} |\xi(E, H_{L_n}, H_{L_n}^\tau)| < \infty \quad \text{for a.e. } E \in \Sigma_{\text{FMB}} \quad (3.23)$$

holds almost surely.

Remarks 3.13. (i) Bounds like (3.23) are unknown for general deterministic continuum Schrödinger operators in $d \geq 2$, and it is not even clear whether they are true. In the case of the pair $-\Delta$ and $-\Delta + W$ with $0 \leq W \in L_c^\infty(\mathbb{R}^d)$, a partial converse is shown in [Kir87]: for any diverging sequence $(L_n)_{n \in \mathbb{N}}$ there exists a dense subset $I \subseteq (0, \infty)$ such that for all $E \in I$

$$\sup_{n \in \mathbb{N}} \xi(-\Delta_{L_n}, -\Delta_{L_n} + W, E) = \infty. \quad (3.24)$$

In contrast, L^p -bounds for finite- or infinite-volume spectral shift functions of deterministic continuum Schrödinger operators are known, see for example [HKNSV06, CHN01, HS02].

(ii) In the case of random Schrödinger operators and for a particular choice of the perturbation, [CHK07b] show that the disorder-averaged finite- and infinite-volume spectral shift functions are locally bounded uniformly in the system size.

The next lemma implies Hölder continuity of the averaged trace of the shift operator in the region of complete localisation.

Lemma 3.14. *Let $I \subset \Sigma_{\text{FMB}}$ be a compact interval and $\alpha \in (0, 1)$. Then there exists a finite constant $C > 0$ such that for all $L > 0$, all $\tau \in [0, 1]$ and all $E, E' \in I$*

$$\mathbb{E} \left[\left\| T(E, H_{(L)}, H_{(L)}^\tau) - T(E', H_{(L)}, H_{(L)}^\tau) \right\|_1 \right] \leq C |E - E'|^\alpha. \quad (3.25)$$

Remark 3.15. Choosing representatives of the spectral shift functions as in Theorem 3.10(i) and (3.21), we infer from Lemma 3.14 that

$$\mathbb{E} \left[\left| \xi(E, H_{(L)}, H_{(L)}^\tau) - \xi(E', H_{(L)}, H_{(L)}^\tau) \right| \right] \leq C |E - E'|^\alpha \quad (3.26)$$

for all $E, E' \in I$.

Lemma 3.14 will be used in the proof of the next theorem which establishes point-wise positivity of the averaged spectral shift function in the region of complete localisation for sufficiently positive perturbations. This will also be useful in the following section on Anderson orthogonality.

Theorem 3.16. *Assume that the Lebesgue density ρ of the single-site distribution, which is specified in (V1), satisfies*

$$\rho_- := \operatorname{ess\,inf}_{x \in [0, 1]} \rho(x) > 0. \quad (3.27)$$

Let $\tau \in (0, 1]$ and assume that the perturbation W and the bump function u_0 satisfy

$$W \geq C u_0 \geq c \mathbf{1}_{B_\delta(x)} \quad (3.28)$$

with constants $c, C, \delta > 0$ and some $x \in \Lambda_1$. Then

$$\mathbb{E} [\xi(E, H, H^\tau)] > 0 \quad \text{for a.e. } E \in \Sigma_{\text{FMB}} \cap \{E' \in \mathbb{R} : \mathcal{N}'(E') > 0\}, \quad (3.29)$$

where \mathcal{N}' is the density of states of H .

Remark 3.17. It is proven in [DGHKM] that under the assumptions of Theorem 3.16 we have

$$\operatorname{ess\,inf}_{E \in I} \mathcal{N}'(E) > 0 \quad (3.30)$$

for every compact interval $I \subset \Sigma_{\text{FMB}} \cap \operatorname{int}(\sigma(H_0) + [0, C_{u,-}])$, where $C_{u,-}$ is the constant from (2.2) and $\operatorname{int}(A)$ stands for the interior of a set $A \in \text{Borel}(\mathbb{R})$. This implies that

$$\mathbb{E}[\xi(E, H, H^\tau)] > 0 \quad \text{for a.e. } E \in \Sigma_{\text{FMB}} \cap \operatorname{int}(\sigma(H_0) + [0, C_{u,-}]). \quad (3.31)$$

3.3. Application 2: Anderson orthogonality. We apply the Schatten-von Neumann-class estimates from Section 3.1 to analyse Anderson orthogonality in the region of complete localisation. Let $\tau > 0$ be fixed for now. The finite-volume operators H_L and H_L^τ have discrete spectrum, and we denote their eigenvalues by

$$\lambda_1^L \leq \lambda_2^L \leq \dots \quad \text{and} \quad \mu_1^L \leq \mu_2^L \leq \dots, \quad (3.32)$$

respectively, ordered by magnitude and repeated according to their multiplicities. The corresponding orthonormal bases of eigenfunctions are $(\varphi_k^L)_{k \in \mathbb{N}}$ and $(\psi_k^L)_{k \in \mathbb{N}}$, respectively. For $N \in \mathbb{N}$ we consider two non-interacting N -Fermion systems in the box Λ_L with single-particle Schrödinger operators H_L and H_L^τ . The corresponding N -particle operators are

$$\mathbf{H}_{N,L}^{(\tau)} := \sum_{j=1}^N \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{(j-1)\text{times}} \otimes H_L^{(\tau)} \otimes \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{(N-j)\text{times}}. \quad (3.33)$$

They act on the totally antisymmetrised tensor product $\mathcal{H}_{N,L} := \bigwedge_{j=1}^N L^2(\Lambda_L)$ with standard scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_{N,L}}$. The respective ground states of $\mathbf{H}_{N,L}$ and $\mathbf{H}_{N,L}^\tau$ are then given by the totally antisymmetrised and normalised tensor products

$$\Phi_{N,L} := \varphi_1^L \wedge \dots \wedge \varphi_N^L \quad \text{and} \quad \Psi_{N,L} := \psi_1^L \wedge \dots \wedge \psi_N^L. \quad (3.34)$$

The *modulus* of their scalar product is given by

$$S_{N,L} := |\langle \Phi_{N,L}, \Psi_{N,L} \rangle_{\mathcal{H}_{N,L}}| = \left| \det \begin{pmatrix} \langle \varphi_1^L, \psi_1^L \rangle & \dots & \langle \varphi_1^L, \psi_N^L \rangle \\ \vdots & & \vdots \\ \langle \varphi_N^L, \psi_1^L \rangle & \dots & \langle \varphi_N^L, \psi_N^L \rangle \end{pmatrix} \right|, \quad (3.35)$$

where the equality follows from the Leibniz formula for determinants, and the scalar product appearing in the matrix entries is the one on the single-particle Hilbert space $L^2(\Lambda_L)$. We note that $S_{N,L}$ depends on the coupling constant τ , but we suppress it in our notation. Our subsequent analysis relies on rewriting $S_{N,L}$ in terms of a Fredholm determinant. We recall that, given an operator $K \in \mathcal{S}^1$ with eigenvalues $(b_n)_{n \in \mathbb{N}}$, listed according to their algebraic multiplicities, the Fredholm determinant of $\mathbb{1} - K$ is defined by

$$\det(\mathbb{1} - K) := \prod_{n \in \mathbb{N}} (1 - b_n). \quad (3.36)$$

It is well defined by the trace-class assumption on K [RS78, Sect. XIII.17]. The following representation of the overlap in terms of Fredholm determinants will be useful.

Lemma 3.18. *Let $N \in \mathbb{N}$, $L > 0$ and define the orthogonal projections*

$$P := P_{N,L} := \sum_{j=1}^N |\varphi_j^L\rangle\langle\varphi_j^L| \quad \text{and} \quad Q := Q_{N,L} := \sum_{k=1}^N |\psi_k^L\rangle\langle\psi_k^L|, \quad (3.37)$$

which we have written down in Dirac notation. Then

$$\begin{aligned} S_{N,L} &= \det(\mathbb{1} - (P - Q)^2)^{1/4} \\ &= \det(\mathbb{1} - P(\mathbb{1} - Q)P)^{1/2} = \det(\mathbb{1} - (\mathbb{1} - P)Q(\mathbb{1} - P))^{1/2}. \end{aligned} \quad (3.38)$$

For fixed $E \in \mathbb{R}$, which will be referred to as the *Fermi energy* in this context, we define the L -dependent particle number

$$N_L(E) := \text{tr}(\mathbb{1}_{(-\infty, E]}(H_L)). \quad (3.39)$$

The limit $L \rightarrow \infty$ is a realisation of the thermodynamic limit with particle density given by the integrated density of states $\lim_{L \rightarrow \infty} N_L(E)/|\Lambda_L| = \mathcal{N}(E)$ of the Hamiltonian H . We define the *finite-volume ground-state overlap*

$$S_L(E) := \begin{cases} S_{N_L(E),L}, & \text{if } N_L(E) \in \mathbb{N}, \\ 1, & \text{if } N_L(E) = 0. \end{cases} \quad (3.40)$$

Following physics terminology, we say that *Anderson orthogonality* occurs at an energy E , if $S_L(E)$ vanishes as $L \rightarrow \infty$.

Note that $P_{N_L(E),L} = \mathbb{1}_{(-\infty, E]}(H_L)$. If $\xi(E, H_L, H_L^\tau) = 0$ holds, then we also have the identity $Q_{N_L(E),L} = \mathbb{1}_{(-\infty, E]}(H_L^\tau)$. In this case we obtain

$$S_L(E) = \det(\mathbb{1} - T(E, H_L, H_L^\tau)^2)^{1/4}. \quad (3.41)$$

This motivates to define the *infinite-volume ground-state overlap* as

$$S(E) := \begin{cases} \det(\mathbb{1} - T(E, H, H^\tau)^2)^{1/4}, & \text{if } T(E, H, H^\tau) \in \mathcal{S}^2, \\ 0, & \text{otherwise.} \end{cases} \quad (3.42)$$

We note that if $E \in \Sigma_{\text{FMB}}$, then $T(E, H, H^\tau) \in \mathcal{S}^p$ almost surely for any $p > 0$ by Corollary 3.4.

Theorem 3.19. (i) *For all $E \in \Sigma_{\text{FMB}} \cap \text{int}(\Sigma)$ the convergence*

$$\lim_{L \rightarrow \infty} \mathbb{E}[S_L(E)] = \mathbb{E}[S(E)] \quad (3.43)$$

holds. If, in addition, $L_n/\ln n \rightarrow \infty$ as $n \rightarrow \infty$, then the pointwise convergence

$$\lim_{n \rightarrow \infty} S_{L_n}(E) = S(E) \quad (3.44)$$

holds almost surely.

(ii) *Let $E \in \Sigma_{\text{FMB}}$. Then*

$$S(E) = 0 \iff 1 \in \sigma(T(E, H, H^\tau)^2) \quad (3.45)$$

holds almost surely.

(iii) *Let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then we have*

$$\liminf_{\tau \downarrow 0} \inf_{E \in I} \mathbb{E}[S(E)] = 1. \quad (3.46)$$

(iv) Suppose that the assumptions of Theorem 3.16 are fulfilled. Then

$$\mathbb{P}[S(E) = 0] > 0 \quad \text{for a.e. } E \in \Sigma_{\text{FMB}} \cap \{E' \in \mathbb{R} : \mathcal{N}'(E') > 0\}, \quad (3.47)$$

where \mathcal{N}' is the density of states of H .

Remarks 3.20. (i) The set of energies for which (3.47) holds is not empty for our model, see Remark 3.17.

(ii) Theorem 3.19(iii) shows that there is no relic of Anderson orthogonality when switching off the perturbation. A similar phenomenon is expected to occur in the large-disorder limit. Indeed, if we suppose that the constant $C \equiv C_\lambda$ in Definition 2.1 vanishes in the strong-disorder limit,

$$\lim_{\lambda \rightarrow \infty} C_\lambda = 0, \quad (3.48)$$

then the proof of Theorem 3.19(iii) shows that we also have

$$\lim_{\lambda \rightarrow \infty} \inf_{E \in I} \mathbb{E}[S(E)] = 1. \quad (3.49)$$

Whether (3.48) holds for alloy-type models in the continuum seems to be unknown [AENSS06, App. A], but it does hold for the discrete Anderson model.

(iii) Theorem 3.19(iv) shows that Anderson orthogonality is not absent almost surely in the region of complete localisation, contrary to established wisdom in the physics literature [GBLA02]. Only very recently physical reasoning and numerical evidence was put forward that Anderson orthogonality does occur in the region of complete localisation [KNS15, DPLDS15].

(iv) For sign-definite perturbations W , the condition (3.45) for Anderson orthogonality to occur in the region of complete localisation is equivalent to a non-vanishing index,

$$S(E) = 0 \iff \theta(E, H, H^\tau) \neq 0 \quad (3.50)$$

almost surely for every $E \in \Sigma_{\text{FMB}}$. This follows from (C.7) in Appendix C.

(v) Let $I \subset \Sigma_{\text{FMB}}$ be compact. Then, Theorem 3.19 (iii) and (iv) imply that there exists $\tau_I > 0$ such that for all $0 < \tau < \tau_I$ and a.e. $E \in I \cap \{E' \in \mathbb{R} : \mathcal{N}'(E') > 0\}$

$$0 < \mathbb{P}[S(E) = 0] < 1. \quad (3.51)$$

Thus, $S(E)$ is not almost surely constant, and both presence and absence of Anderson orthogonality occur with positive probability for energies in the region of complete localisation.

(vi) Let W be a sign-definite perturbation and let $E \in \Sigma_{\text{FMB}} \cap \text{int}(\Sigma)$ such that $\theta(E, H, H^\tau) = 0$, that is, Anderson orthogonality cannot occur at this energy according to (3.50). In this case our proof of Theorem 3.19(i) provides exponential speed of convergence of the overlap in (3.44): we conclude from (6.13) and (3.9) that, given a sequence of lengths with $L_n/\ln n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a constant $c > 0$ such that

$$\lim_{n \rightarrow \infty} e^{cL_n} |S_{L_n}(E) - S(E)| = 0 \quad (3.52)$$

holds almost surely. According to the physical arguments in [DPLDS15], exponential speed of convergence occurs with positive probability for $E \in \Sigma_{\text{FMB}}$ even if $\theta(E, H, H^\tau) \neq 0$. More precisely, it is argued there that $\mathbb{E}[\ln S_L(E)] \sim -L$ for large L . This phenomenon is dubbed *statistical Anderson orthogonality* in [DPLDS15] to

distinguish it from the usual algebraic decay for energies in the scattering regime, see also the next remark.

Similarly to Corollary 3.8, Theorem 3.19(i) and (iii) offer a way to detect spectral phases of random Schrödinger operators other than the region of complete localisation.

Corollary 3.21. *Let $E \in \text{int}(\Sigma)$. Assume there exists a null sequence of coupling constants $(\tau_k)_{k \in \mathbb{N}} \subset (0, 1)$ and a sequence of lengths $(L_n)_{n \in \mathbb{N}}$ with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$ such that for every $k \in \mathbb{N}$*

$$\lim_{n \rightarrow \infty} S_{L_n}(E) = 0 \quad \text{almost surely,} \quad (3.53)$$

then $E \notin \Sigma_{\text{FMB}}$.

Remark 3.22. The criterion of Corollary 3.21 should be compared to the behaviour of the overlap in the (hypothetical) conducting phase, provided $W \geq 0$: suppose we knew there exists a spectral interval $J \subset \Sigma$ such that H has absolutely continuous spectrum in J almost surely. Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of lengths such that $L_n / e^{n^\alpha} \rightarrow \infty$ as $n \rightarrow \infty$ for some $\alpha > 1$. Then, Theorem 2.2 in [GKMO16], see also [GKM14], applies realisationwise to the operators H and H^τ for any $\tau \in (0, 1]$, and we infer that almost surely and for a.e. energy $E \in J$

$$S_{L_n}(E) \leq L_n^{-\gamma(E)/2 + o(L_n^0)} \quad \text{as } n \rightarrow \infty. \quad (3.54)$$

The decay exponent

$$\gamma(E) := \frac{1}{2\pi^2} \left\| \arcsin \left| \frac{\mathbb{1} - \mathbb{S}_E}{2} \right| \right\|_2^2 \quad (3.55)$$

relates to the energy-dependent scattering matrix \mathbb{S}_E and is strictly positive if the perturbation W causes non-trivial scattering at energy E . This demonstrates that the criterion in Corollary 3.21 is potentially useful because its assumption holds in this case. We remark that the superexponential growth of the length scales $(L_n)_{n \in \mathbb{N}}$ avoids the necessity of passing to a subsequence in (3.54) as is done in [GKMO16, Theorem 2.2], because it implies a sufficiently fast L^1 -convergence in the proof of [GKMO16, Lemma 3.3(i)] so that subsequences can be avoided in that lemma.

4. Proofs of the results from Section 3.1

4.1. Helffer-Sjöstrand formula. Our analysis relies on an application of the (Dynkin-) Helffer-Sjöstrand formula which works for compactly supported functions of bounded variation. Let \mathcal{H} be a Hilbert space, K a self-adjoint operator on \mathcal{H} and $f : \mathbb{R} \rightarrow \mathbb{C}$ a sufficiently regular function. According to the Helffer-Sjöstrand formula, see e.g. [Dav95], there exists a complex Borel measure ζ_f on \mathbb{R}^2 such that

$$f(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_f(x, y) R_{x+iy}(K). \quad (4.1)$$

The smoothness of f determines how fast the measure ζ_f vanishes in the vicinity of the horizontal axis $y = 0$. This has to compensate the potential divergence of the resolvent as $|y| \rightarrow 0$. For instance, if $f \in C_c^2(\mathbb{R})$, then ζ_f can be chosen as $d\zeta_f(x, y) = dx dy \tilde{f}(x, y)$ with

$$\tilde{f}(x, y) := (\partial_x + i\partial_y) ((f(x) + iyf'(x))\Xi(x, y)). \quad (4.2)$$

Here, $\Xi \in C_c^\infty(\mathbb{R}^2)$ is a cutoff function with $\Xi \equiv 1$ in a neighbourhood N_f of $\text{supp}(f) \times \{0\} \subset \mathbb{R}^2$. In particular, the integral in (4.1) is well defined in this case. The Helffer-Sjöstrand formula (4.1) implies

$$Af(K)B = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_f(x, y) AR_{x+iy}(K)B \quad (4.3)$$

for all bounded operators A, B on \mathcal{H} . Now, it may be the case that the product $AR_{x+iy}(K)B$ has a less severe divergence as $|y| \rightarrow 0$ than the resolvent alone. In such cases, the right-hand side of (4.3) typically is well defined for functions f which are less regular than what is needed for (4.1). This is made precise in the next lemma.

For its formulation we consider functions $f \in \text{BV}_c(\mathbb{R})$. Given such f , we choose a cutoff function $\Xi \in C_c^\infty(\mathbb{R}^2)$ with $\Xi \equiv 1$ in a neighbourhood N_f of $\text{supp}(f) \times \{0\} \subset \mathbb{R}^2$. We then define the complex Borel measure ζ_f on \mathbb{R}^2 by

$$d\zeta_f(x, y) := df(x) dy \Xi(x, y) + dx dy f(x) (\partial_x + i\partial_y) \Xi(x, y), \quad (4.4)$$

where df denotes Lebesgue-Stieltjes integration with respect to f . We write $|\zeta_f|$ for the total variation measure of ζ_f .

Lemma 4.1. *Let $f \in \text{BV}_c(\mathbb{R})$ and ζ_f as in (4.4). Let K be a self-adjoint operator and let A, B be bounded operators on the Hilbert space \mathcal{H} . If*

$$\int_{\mathbb{R}^2} d|\zeta_f|(x, y) \|AR_{x+iy}(K)B\| < \infty, \quad (4.5)$$

then

$$Af(K)B = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_f(x, y) AR_{x+iy}(K)B \quad (4.6)$$

holds, where the right-hand side is a Bochner integral with respect to the operator norm.

Remark 4.2. Lemma 4.1 can be extended to appropriate Besov spaces, $B_{p,q}^s$, $1 \leq p, q \leq \infty$ and $0 < s < 1$, by using Dynkin's characterization of Besov spaces in terms of quasi-analytic extensions [Dyn83].

Before we prove the lemma, we apply it to spatially localised functions of random Schrödinger operators which obey the a priori estimate (2.7).

Corollary 4.3. *Let $G \subseteq \mathbb{R}^d$ be open, $a, b \in G$ and let H_G^τ be defined as in (2.1) and (2.5). Then, for $f \in \text{BV}_c(\mathbb{R})$, the equality*

$$\chi_a f(H_G^\tau) \chi_b = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_f(x, y) \chi_a R_{x+iy}(H_G^\tau) \chi_b \quad (4.7)$$

holds almost surely.

PROOF. We verify that (4.5) holds almost surely. There exists $\delta > 0$ (independent of f) such that $\text{supp}(\Xi) \subset \mathbb{R} \times [-\delta, \delta]$. For fixed $0 < s < 1$, we use the deterministic norm bound $\|R_z(H_G^\tau)\|^{1-s} \leq 1/|\text{Im}z|^{1-s}$, $z \in \mathbb{C} \setminus \mathbb{R}$, and estimate

$$\mathbb{E} \left[\int_{\mathbb{R}^2} d|\zeta_f|(x, y) \|\chi_a R_{x+iy}(H_G^\tau) \chi_b\| \right] \leq C_s \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s}}, \quad (4.8)$$

where

$$C_s := \sup_{\substack{(E, \eta) \in \text{supp}(\zeta_f) \\ \eta \neq 0}} \mathbb{E} [\|\chi_a R_{E+i\eta}(H_G^T) \chi_b\|^s] < \infty \quad (4.9)$$

is finite by the a priori estimate (2.7). The claim then follows from

$$\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s}} \leq \frac{2}{s} \delta^s (\text{TV}(f) \|\Xi\|_\infty + 2\|f\|_1 \|\nabla \Xi\|_\infty). \quad (4.10)$$

□

PROOF OF LEMMA 4.1. The first part of the proof closely follows the standard proof of the Helffer-Sjöstrand formula. We note that assumption (4.5) implies that the right-hand side of (4.6) is well defined as a Bochner integral with respect to the operator norm. To show (4.6) we let $\varepsilon > 0$ and introduce the closed horizontal strip $\mathcal{C}_\varepsilon := \mathbb{R} \times [-\varepsilon, \varepsilon]$ in \mathbb{R}^2 . We split the integral according to

$$\begin{aligned} \int_{\mathbb{R}^2} d\zeta_f(x, y) A R_{x+iy}(K) B &= \int_{\mathcal{C}_\varepsilon} d\zeta_f(x, y) A R_{x+iy}(K) B \\ &\quad + \int_{\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon} d\zeta_f(x, y) A R_{x+iy}(K) B \\ &=: I_1^\varepsilon + I_2^\varepsilon. \end{aligned} \quad (4.11)$$

Because of (4.5), dominated convergence yields $\lim_{\varepsilon \downarrow 0} \|I_1^\varepsilon\| = 0$. As for the second integral, we note that $C \mapsto ACB$ is a norm-continuous linear map on the Banach space of bounded linear operators. The properties of the Bochner integral then imply

$$I_2^\varepsilon = A \left(\int_{\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon} d\zeta_f(x, y) R_{x+iy}(K) \right) B. \quad (4.12)$$

Here, the right-hand side is well defined because $\text{supp}(\zeta_f)$ is compact and the norm of the resolvent is uniformly bounded on $\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon$. Now we choose $\varepsilon > 0$ so small that $\Xi \equiv 1$ on $\text{supp}(f) \times [-\varepsilon, \varepsilon]$. It follows from Fubini's theorem that $I_2^\varepsilon = 2\pi A f_\varepsilon(K) B$ with

$$\begin{aligned} f_\varepsilon(\lambda) &:= \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon} \frac{d\zeta_f(x, y)}{\lambda - x - iy} = \frac{i}{2\pi} \int_{\mathbb{R}} dx \left(-\frac{f(x) \Xi(x, \varepsilon)}{\lambda - x - i\varepsilon} + \frac{f(x) \Xi(x, -\varepsilon)}{\lambda - x + i\varepsilon} \right) \\ &= \int_{\mathbb{R}} dx f(x) \frac{\varepsilon/\pi}{(\lambda - x)^2 + \varepsilon^2} = \int_{\mathbb{R}} dx \frac{1/\pi}{x^2 + 1} f(\lambda + \varepsilon x) \end{aligned} \quad (4.13)$$

for $\lambda \in \mathbb{R}$, where the first equality relies on integration by parts and holomorphy $(\partial_x + i\partial_y)(\lambda - x - iy)^{-1} = 0$ on $\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon$. The second equality follows from $\Xi(x, \varepsilon) = \Xi(x, -\varepsilon) = 1$ for all $x \in \text{supp}(f)$.

The second part of the proof deals with the problem that discontinuity points of f challenge the convergence of I_2^ε as $\varepsilon \downarrow 0$. However, the regularity condition (4.5) ensures that they form only a null set of the relevant spectral measures and, thus, weak convergence still holds. To see this, let $\varphi, \psi \in \mathcal{H}$ and define the complex spectral measure $\mu_{\varphi, \psi} := \langle \varphi, A \mathbb{1}_\cdot(K) B \psi \rangle$ of K . The functional calculus, (4.13) and dominated convergence imply

$$\lim_{\varepsilon \downarrow 0} \langle \varphi, A f_\varepsilon(K) B \psi \rangle = \int_{\mathbb{R}} d\mu_{\varphi, \psi}(\lambda) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} dx \frac{1/\pi}{x^2 + 1} f(\lambda + \varepsilon x). \quad (4.14)$$

We prove below that the set of discontinuity points of f is a $\mu_{\varphi,\psi}$ -null set. Using this, another application of dominated convergence in (4.14) yields $\lim_{\varepsilon \downarrow 0} I_2^\varepsilon = 2\pi A f(K) B$ weakly, and the lemma follows.

It remains to prove that f is continuous $\mu_{\varphi,\psi}$ -almost everywhere. Without loss of generality, we assume $\|\varphi\| = \|\psi\| = 1$. Since $f \in \text{BV}_c(\mathbb{R})$, it has left and right limits at all points. Hence, the set $\mathcal{U} := \{\lambda \in \mathbb{R} : f \text{ not continuous in } \lambda\}$ consists of jump discontinuities only. Moreover, \mathcal{U} is countable so that

$$\mu_{\varphi,\psi}(\mathcal{U}) = \sum_{\lambda \in \mathcal{U}} \mu_{\varphi,\psi}(\{\lambda\}). \quad (4.15)$$

We fix an arbitrary $\lambda \in \mathcal{U}$ and set $\delta f_\lambda := \lim_{\varepsilon \downarrow 0} [f(\lambda + \varepsilon) - f(\lambda - \varepsilon)] \neq 0$. We choose $y_0 > 0$ small enough such that $\Xi(\lambda, y) = 1$ whenever $|y| \leq y_0$. Assumption (4.5) implies

$$\begin{aligned} \infty &> \int_{\mathbb{R}^2} d|\zeta_f|(x, y) \|A R_{x+iy}(K) B\| \geq |\delta f_\lambda| \int_{-y_0}^{y_0} dy \|A R_{\lambda+iy}(K) B\| \\ &\geq |\delta f_\lambda| \int_{-y_0}^{y_0} \frac{dy}{|y|} |h(y)|, \end{aligned} \quad (4.16)$$

where $h(y) := \int_{\mathbb{R}} d\mu_{\varphi,\psi}(\lambda') y / (\lambda' - \lambda - iy)$. Dominated convergence implies that $\mathbb{R} \ni y \mapsto h(y)$ is continuous and that $\lim_{y \rightarrow 0} h(y) = i \mu_{\varphi,\psi}(\{\lambda\})$. Now, we assume that $\mu_{\varphi,\psi}(\{\lambda\}) \neq 0$. Then there exists $0 < y_1 \leq y_0$ such that $|h(y)| \geq |\mu_{\varphi,\psi}(\{\lambda\})|/2$ whenever $|y| \leq y_1$, and we conclude that (4.16) yields a contradiction. Therefore we must have $\mu_{\varphi,\psi}(\{\lambda\}) = 0$, and (4.15) implies the desired continuity of f . \square

4.2. Proof of Theorem 3.1 and Corollary 3.4. We introduce a lower bound of the spectra of all considered random Schrödinger operators,

$$\begin{aligned} E_0 &:= \min \left\{ \inf_{x \in \mathbb{R}^d} \{V_0(x)\}, \inf_{x \in \mathbb{R}^d} \{V_0(x) + W(x)\} \right\} \\ &\leq \min \left\{ \min\{\sigma(H_G)\}, \min\{\sigma(H_G^\tau)\} \right\}. \end{aligned} \quad (4.17)$$

Here, the inequality holds almost surely for all $\tau \in [0, 1]$ and all $G \subseteq \mathbb{R}^d$ open.

We begin with an operator-norm version of Theorem 3.1. Given a function $f \in \text{BV}(\mathbb{R})$, we use the notation $\text{supp}(f')$ for the support of the complex measure defined by Lebesgue-Stieltjes integration with respect to f .

Theorem 4.4. *Fix $0 < s < 1$ and a compact set $S \subset \Sigma_{\text{FMB}}$. Then there exist finite constants $C, \mu > 0$ such that for all functions $f \in \text{BV}_c(\mathbb{R})$ with $\text{supp}(f') \subseteq S$, for all $\tau \in [0, 1]$, all open $G \subseteq \mathbb{R}^d$ and all $a, b \in \mathbb{R}^d$ we have*

$$\mathbb{E} \left[\left\| \chi_a(f(H_G) - f(H_G^\tau)) \chi_b \right\| \right] \leq C |\tau|^{s/2} (\|f\|_1 + \text{TV}(f)) e^{-\mu(|a|+|b|)}. \quad (4.18)$$

PROOF. We fix $f \in \text{BV}_c(\mathbb{R})$ such that $\text{supp}(f') \subseteq S$. Let $S_- := \inf S$, $S_+ := \sup S$ and observe $\text{supp}(f) \subseteq [S_-, S_+]$. We choose a cutoff function $\Xi = \Xi_f$ subject to

- (P1) $\Xi \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq \Xi \leq 1$ and $\|\partial_x \Xi\|_\infty, \|\partial_y \Xi\|_\infty \leq 3$,
- (P2) $\text{supp}(\Xi) \subseteq [S_- - 1, S_+ + 1] \times [-1, 1]$,
- (P3) $\Xi \equiv 1$ on $[S_- - 1/2, S_+ + 1/2] \times [-1/2, 1/2]$.

Now fix $G \subseteq \mathbb{R}^d$ open and $a, b \in \mathbb{R}^d$. Let ζ_h be the complex Borel measure defined in (4.4). We apply Corollary 4.3 to the operators $f(H_G)$ and $f(H_G^\tau)$. This gives the estimate

$$\begin{aligned} & \mathbb{E} [\|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|] \\ & \leq \frac{1}{2\pi} \mathbb{E} \left[\int_{\mathbb{R}^2} d|\zeta_f|(x, y) \|\chi_a(R_{x+iy}(H_G) - R_{x+iy}(H_G^\tau))\chi_b\| \right] \\ & \leq \frac{2^{1-s/2}}{2\pi} \mathbb{E} \left[\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s/2}} \|\chi_a(R_{x+iy}(H_G) - R_{x+iy}(H_G^\tau))\chi_b\|^{s/2} \right], \end{aligned} \quad (4.19)$$

where, in the last step, we chose some $0 < s < 1$ and used the norm bound $\|R_z(H_G^{(\tau)})\|^{1-s/2} \leq 1/|\operatorname{Im} z|^{1-s/2}$ for the resolvent of a self-adjoint operator. We introduce the subset of lattice points

$$\Gamma_W := \{n \in \mathbb{Z}^d : \operatorname{dist}(n, \operatorname{supp}(W)) \leq 1/2\} \quad (4.20)$$

needed as centres to cover $\operatorname{supp}(W)$ by closed unit cubes and deduce from the resolvent equation and the Cauchy-Schwarz inequality

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s/2}} \|\chi_a(R_{x+iy}(H_G) - R_{x+iy}(H_G^\tau))\chi_b\|^{s/2} \right] \\ & \leq \|\tau W\|_\infty^{s/2} \sum_{c \in \Gamma_W} \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s/2}} \\ & \quad \times \mathbb{E} \left[\|\chi_a R_{x+iy}(H_G)\chi_c\|^{s/2} \|\chi_c R_{x+iy}(H_G^\tau)\chi_b\|^{s/2} \right] \\ & \leq \|\tau W\|_\infty^{s/2} \sum_{c \in \Gamma_W} \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s/2}} \\ & \quad \times \mathbb{E} [\|\chi_a R_{x+iy}(H_G)\chi_c\|^s]^{1/2} \mathbb{E} [\|\chi_c R_{x+iy}(H_G^\tau)\chi_b\|^s]^{1/2}. \end{aligned} \quad (4.21)$$

We infer from (4.4), (P2) and (P3) that

$$\operatorname{supp}(\zeta_f) \subseteq \left(\operatorname{supp}(f') \times [-1, 1] \right) \cup \left([S_-, S_+] \times ([-1, -1/2] \cup [1/2, 1]) \right) =: Z_1 \cup Z_2. \quad (4.22)$$

On the set Z_2 we estimate the right-hand side of (4.21) by the Combes-Thomas estimate stated in [GK03, Cor. 1] for deterministic Schrödinger operators. Even though this result is formulated for Schrödinger operators on $L^2(\mathbb{R}^d)$, the argument extends to Schrödinger operators on $L^2(G)$ for arbitrary $G \subseteq \mathbb{R}^d$ open – see also [She14]. Thus, there exist constants $C_0, \mu_0 > 0$, which are independent of $G \subseteq \mathbb{R}^d$, $\tau \in [0, 1]$ and $\omega \in \Omega$, such that

$$\|\chi_a R_z(H_G^\tau)\chi_b\| \leq C_0 e^{-\mu_0|z|^{-1}|a-b|} \quad (4.23)$$

for all $z \in \mathbb{C}$ subject to $\operatorname{dist}(z, \sigma(H_G^\tau)) \geq 1/2$. For $(x, y) \in Z_2$ we have

$$|x + iy| \leq |x| + |y| \leq \max\{|S_-|, |S_+|\} + 1 =: C_S \quad (4.24)$$

and therefore we obtain finite constants $C_1 > 0$ and $\mu_1 := 2s\mu_0/C_S > 0$ such that

$$\sup_{(x, y) \in Z_2} \left\{ \mathbb{E} [\|\chi_a R_{x+iy}(H_G)\chi_c\|^s] \mathbb{E} [\|\chi_c R_{x+iy}(H_G^\tau)\chi_b\|^s] \right\} \leq C_1 e^{-\mu_1(|a-c|+|c-b|)}. \quad (4.25)$$

On the set Z_1 we use the fractional-moment bounds (2.6) for H and H^τ , which can be applied because of $\text{supp}(f') \subseteq \Sigma_{\text{FMB}}$ and Remark 2.2(iii). Hence, there exist finite constants $C_2 > 0$ and $\mu_2 > 0$ such that

$$\sup_{(x,y) \in Z_1} \left\{ \mathbb{E}[\|\chi_a R_{x+iy}(H_G) \chi_c\|^s] \mathbb{E}[\|\chi_c R_{x+iy}(H_G^\tau) \chi_b\|^s] \right\} \leq C_2 e^{-\mu_2(|a-c|+|c-b|)}. \quad (4.26)$$

Collecting the estimates in (4.19), (4.21), (4.25) and (4.26), we obtain finite constants $C_3 > 0$ and $\mu := \min\{\mu_1, \mu_2\}/2 > 0$, which depend on s and S but are independent of G and $\tau \in [0, 1]$ such that

$$\begin{aligned} \mathbb{E}[\|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|] &\leq C_3 \|\tau W\|_\infty^{s/2} \sum_{c \in \Gamma_W} e^{-\mu(|a-c|+|c-b|)} \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x, y)}{|y|^{1-s/2}} \\ &\leq C |\tau|^{s/2} (\|f\|_1 + \text{TV}(f)) e^{-\mu(|a|+|b|)}. \end{aligned} \quad (4.27)$$

To obtain the last inequality, we applied (4.10) and the estimate

$$\sum_{c \in \Gamma_W} e^{-\mu(|a-c|+|c-b|)} \leq e^{-\mu(|a|+|b|)} \sum_{c \in \Gamma_W} e^{2\mu|c|}. \quad (4.28)$$

□

PROOF OF THEOREM 3.1. We set $I_- := \min I$, $I_+ := \max I$ and assume without restriction that $I_+ \geq E_0$, the lower bound (4.17) for the spectrum of H_G^τ (otherwise the statement is trivial). We fix $G \subseteq \mathbb{R}^d$ open and $a, b \in \mathbb{R}^d$. Since $\|A\|_p \leq \|A\|_1$ for any Schatten norm with $p \geq 1$, we restrict ourselves to $0 < p \leq 1$.

Let $f \in \mathcal{F}_I$. As we want to apply Theorem 4.4, which requires functions of compact support, we introduce $h := f \mathbf{1}_{[E_0-1, \infty)}$. Then we have $h \in \text{BV}_c(\mathbb{R})$, $\text{TV}(h) \leq 2 \text{TV}(f)$,

$$f(H_G^\tau) = h(H_G^\tau) \quad (4.29)$$

for every $\tau \in [0, 1]$ and $\text{supp}(h') \subseteq I \cup \{E_0 - 1\} \subset \Sigma_{\text{FMB}}$. The last inclusion holds by the definition of E_0 and the Combes-Thomas estimate [GK03, Cor. 1], which extends to finite-volume operators.

Because of (4.29) and Lemma B.1 we obtain for any $0 < r < p$ that

$$\begin{aligned} \|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|_p &= \|\chi_a(h(H_G) - h(H_G^\tau))\chi_b\|_p \\ &\leq \|\chi_a(h(H_G) - h(H_G^\tau))\chi_b\|^{r/p} \|\chi_a(h(H_G) - h(H_G^\tau))\chi_b\|_{p-r}^{1-r/p}. \end{aligned} \quad (4.30)$$

The adapted triangle inequality (3.2) and the deterministic a priori estimate from Lemma B.2 imply

$$\|\chi_a(h(H_G) - h(H_G^\tau))\chi_b\|_{p'}^{p'} \leq \|\chi_a h(H_G) \chi_b\|_{p'}^{p'} + \|\chi_a h(H_G^\tau) \chi_b\|_{p'}^{p'} \leq C_{p'} \|h\|_\infty^{p'} \quad (4.31)$$

for every $0 < p' \leq 1$, where $C_{p'}$ depends also on I_+ , but is uniform in the disorder, G and $\tau \in [0, 1]$, and independent of h . We apply (4.31) with $p' = p - r$ to estimate the expectation of (4.30) by

$$\mathbb{E}[\|\chi_a(f(H_G) - f(H_G^\tau))\chi_b\|_p] \leq C_{p-r}^{1/p} \|h\|_\infty^{1-s_1} \mathbb{E}[\|\chi_a(h(H_G) - h(H_G^\tau))\chi_b\|_p]^{s_1}. \quad (4.32)$$

Here, we introduced $s_1 := r/p < 1$ and also used Jensen's inequality.

Now, we choose $0 < s_2 < 1$ and apply Theorem 4.4 with $S = I \cup \{E_0 - 1\}$ to the expectation on the right-hand side of (4.32). This yields finite constants $C_1, \mu_1 > 0$ (which depend on s_2 and I) such that

$$\mathbb{E} \left[\left\| \chi_a (h(H_G) - h(H_G^\tau)) \chi_b \right\| \right] \leq C_1 |\tau|^{s_2/2} (I_+ - E_0 + 2) \text{TV}(f) e^{-\mu_1(|a|+|b|)}, \quad (4.33)$$

where we used

$$\|h\|_1 + \text{TV}(h) \leq 2(I_+ - E_0 + 2) \text{TV}(f). \quad (4.34)$$

Inserting (4.33) into (4.32) and observing

$$\|h\|_\infty \leq \text{TV}(h) \leq 2 \text{TV}(f), \quad (4.35)$$

we obtain

$$\mathbb{E} \left[\left\| \chi_a (f(H_G) - f(H_G^\tau)) \chi_b \right\|_p \right] \leq C |\tau|^{s_1 s_2/2} \text{TV}(f) e^{-s_1 \mu_1(|a|+|b|)} \quad (4.36)$$

with a suitable finite constant $C > 0$. Since $s_1, s_2 \in (0, 1)$ are both arbitrary, the claim follows with $s := s_1 s_2$. \square

PROOF OF COROLLARY 3.4. The (quasi-) norm estimates

$$\mathbb{E} [|X|^{q_1}]^{1/q_1} \leq \mathbb{E} [|X|^{q_2}]^{1/q_2} \quad \text{and} \quad \|A\|_{p_1} \leq \|A\|_{p_2} \quad (4.37)$$

for random variables X and Schatten-class operators A hold for all $0 < p_2 \leq p_1 < \infty$ and all $0 < q_1 \leq q_2 < \infty$. Thus, we assume without loss of generality that $p \leq 1$, $q \geq 1$ such that $k := q/p \in \mathbb{N}$.

For $f \in \mathcal{F}_I$ and $G \subseteq \mathbb{R}^d$ open we abbreviate

$$T_f := f(H_G) - f(H_G^\tau). \quad (4.38)$$

Because of the adapted triangle inequality (3.2) for $p \leq 1$ we estimate

$$\|T_f\|_p^p \leq \sum_{a,b \in \mathbb{Z}^d} \|\chi_a T_f \chi_b\|_p^p. \quad (4.39)$$

A k -fold application of (4.39) yields

$$\mathbb{E} [\|T_f\|_p^q] = \mathbb{E} [\|T_f\|_p^{pk}] \leq \sum_{\substack{a_1, \dots, a_k \in \mathbb{Z}^d \\ b_1, \dots, b_k \in \mathbb{Z}^d}} \mathbb{E} \left[\prod_{l=1}^k \|\chi_{a_l} T_f \chi_{b_l}\|_p^p \right]. \quad (4.40)$$

Next, we apply Hölder's inequality to the expectation in (4.40) and obtain

$$\mathbb{E} [\|T_f\|_p^q] \leq \sum_{\substack{a_1, \dots, a_k \in \mathbb{Z}^d \\ b_1, \dots, b_k \in \mathbb{Z}^d}} \prod_{l=1}^k \mathbb{E} [\|\chi_{a_l} T_f \chi_{b_l}\|_p^{pk}]^{1/k}. \quad (4.41)$$

Now, we choose $0 < s < 1$. Theorem 3.1 implies the existence of finite constants $C, \mu > 0$ (depending only on p, I, s) such that

$$\mathbb{E} [\|\chi_{a_l} T_f \chi_{b_l}\|_p] \leq C \tau^{s/2} \text{TV}(f) e^{-\mu(|a_l|+|b_l|)} \quad (4.42)$$

for all $a_l, b_l \in \mathbb{Z}^d$. The deterministic a priori estimate from Lemma B.2 provides the existence of a finite constant C_1 (depending only on p, I, s , but not on ω) such that

$$\|\chi_{a_l} T_f \chi_{b_l}\|_p^{pk-1} \leq C_1 \text{TV}(f)^{pk-1}. \quad (4.43)$$

We thus obtain

$$\mathbb{E}[\|\chi_{a_l} T_f \chi_{b_l}\|_p^{pk}] \leq CC_1 \tau^{s/2} \text{TV}(f)^{pk} e^{-\mu(|a_l|+|b_l|)} \quad (4.44)$$

for all $a_l, b_l \in \mathbb{Z}^d$. Estimating the right-hand side of (4.41) by (4.44) and using $q = pk$, we arrive at

$$\mathbb{E}[\|T_f\|_p^q] \leq C_2 \tau^{s/2} \text{TV}(f)^q, \quad (4.45)$$

where the finite constant C_2 depends on p, I, s . In particular, the constant C_τ in the statement vanishes algebraically as $\tau \downarrow 0$. \square

4.3. Proof of Theorem 3.2 and Theorem 3.6.

PROOF OF THEOREM 3.2. We will follow the strategy in the proofs of Theorems 4.4 and 3.1 and use the notation introduced there. As it is done there, we assume without restriction that $I_+ \geq E_0$ and that $0 < p \leq 1$. Moreover, we restrict ourselves to the case $b \in \mathbb{R}^d$ with $Q_b \cap G \neq \emptyset$.

Let $a \in \mathbb{R}^d$, $\tau \in [0, 1]$, $f \in \mathcal{F}_I$ and define again the truncation $h := f \mathbf{1}_{[E_0-1, \infty)}$. We write ζ_h for the complex measure defined as in (4.4) with a cutoff function $\Xi = \Xi_h$ that satisfies (P1) – (P3), where f is replaced by h . Proceeding along the lines of (4.32) and (4.19), we obtain for any $s, s' \in (0, 1)$

$$\begin{aligned} \mathbb{E}[\|\chi_a(f(H_G^\tau) - f(H_{\tilde{G}}^\tau))\chi_b\|_p] &= \mathbb{E}[\|\chi_a(h(H_G^\tau) - h(H_{\tilde{G}}^\tau))\chi_b\|_p] \\ &\leq C_1 \|h\|_\infty^{1-s'} \mathbb{E}[\|\chi_a(h(H_G^\tau) - h(H_{\tilde{G}}^\tau))\chi_b\|]^{s'} \\ &\leq C_2 \|h\|_\infty^{1-s'} \mathbb{E}\left[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x, y)}{|y|^{1-s/2}} \|\chi_a(R_{x+iy}(H_G^\tau) - R_{x+iy}(H_{\tilde{G}}^\tau))\chi_b\|^{s/2}\right]^{s'} \end{aligned} \quad (4.46)$$

with finite constants $C_1 = C_{1,p,s',I_+} > 0$ and $C_2 = C_{2,p,s,s',I_+} > 0$.

Case 1. We assume $\text{dist}(a, \partial G) > 1$ and $\text{dist}(b, \partial G) > 1$. Hence, we have $Q_b \subset G$ in this case. We apply the geometric resolvent inequality, see e.g. [Sto01, Lemma 2.5.2], to the operator norm in the last line of (4.46). Even though it is only stated for boxes there, the key estimate, [Sto01, Lemma 2.5.3], covers our setup. Hereby we use the assumption $\text{dist}(\partial G, \partial \tilde{G}) > 1$. We obtain in analogy to [DGHKM, Eq. (4.9)]

$$\|\chi_a(R_z(H_G^\tau) - R_z(H_{\tilde{G}}^\tau))\chi_b\| \leq C_3 \sum_{c \in \delta G^\#} \|\chi_a R_z(H_{\tilde{G}}^\tau) \chi_c\| \|\mathbf{1}_{\Lambda_2(c)} R_z(H_G^\tau) \chi_b\|, \quad (4.47)$$

where $\Lambda_2(c) := c + \Lambda_2$ and

$$\delta G^\# := \{n \in \mathbb{Z}^d : \text{dist}(n, \partial G) \leq 1\} \quad (4.48)$$

denotes the set of lattice points needed as centres to cover a strip of width 1 around the boundary ∂G of $G \subset \mathbb{R}^d$ by unit cubes. The constant C_3 is uniform in $z \in \mathbb{C}$ on each compact subset of \mathbb{C} . It is also uniform in $a \in \mathbb{R}^d$ and $b \in G$ with $\text{dist}(b, \partial G) >$

1. Inserting this into (4.46) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x, y)}{|y|^{1-s/2}} \left\| \chi_a(R_{x+iy}(H_G^\tau) - R_{x+iy}(H_{\tilde{G}}^\tau)) \chi_b \right\|^{s/2} \right] \\ & \leq C_4 \sum_{c \in \delta G^\#} \int_{\mathbb{R}^2} \frac{d|\zeta_h|(x, y)}{|y|^{1-s/2}} \mathbb{E} \left[\left\| \chi_a R_{x+iy}(H_G^\tau) \chi_c \right\|^s \right]^{1/2} \\ & \quad \times \mathbb{E} \left[\left\| \mathbf{1}_{\Lambda_2(c)} R_{x+iy}(H_G^\tau) \chi_b \right\|^s \right]^{1/2} \end{aligned} \quad (4.49)$$

with a constant $C_4 = C_{4,s,I_+}$. We decompose the support of ζ_h as in (4.22) and treat the product of the expectations on Z_1 with the fractional-moment estimate as in (4.26) and on Z_2 with the Combes-Thomas estimate as in and (4.25) (with H_G replaced by $H_{\tilde{G}}^\tau$ in both estimates). The remaining integral is estimated as in (4.10). We then arrive at

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x, y)}{|y|^{1-s/2}} \left\| \chi_a(R_{x+iy}(H_G^\tau) - R_{x+iy}(H_{\tilde{G}}^\tau)) \chi_b \right\|^{s/2} \right] \\ & \leq C_5 \sum_{c \in \delta G^\#} e^{-\mu(|a-c|+|c-b|)} (\|h\|_1 + \text{TV}(h)) \\ & \leq C_6 \text{TV}(f) e^{-(\mu/2)[\text{dist}(a, \partial G) + \text{dist}(b, \partial G)]} \end{aligned} \quad (4.50)$$

with finite constants $C_5, C_6, \mu > 0$, all depending only on s and I . In the last step we used (4.34), the estimate

$$\sum_{c \in \delta G^\#} e^{-\mu(|a-c|+|c-b|)} \leq e^{-(\mu/2)[\text{dist}(a, \partial G) + \text{dist}(b, \partial G) - 2]} \sum_{c \in \mathbb{Z}^d} e^{-(\mu/2)(|a-c|+|c-b|)} \quad (4.51)$$

and that the sum on the right-hand side of (4.51) is bounded from above uniformly in $a, b \in \mathbb{R}^d$ by, e.g., the Cauchy-Schwarz inequality. Now, the claim follows upon inserting (4.50) into (4.46) and observing (4.35). This finishes Case 1.

Case 2. We assume $\text{dist}(a, \partial G) \leq 1$ or $\text{dist}(b, \partial G) \leq 1$. Hence, we have

$$|a - b| \geq \max \{ \text{dist}(a, \partial G), \text{dist}(b, \partial G) \} - 1 \quad (4.52)$$

in this case. We estimate the operator norm on the right-hand side of (4.46) by the triangle inequality and each of the resulting two terms by the fractional-moment estimate (2.6), see also Remark 2.2(iii). The remaining integral is again estimated by (4.10) and (4.34), and we obtain

$$\mathbb{E} \left[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x, y)}{|y|^{1-s/2}} \left\| \chi_a(R_{x+iy}(H_G^\tau) - R_{x+iy}(H_{\tilde{G}}^\tau)) \chi_b \right\|^{s/2} \right] \leq C_7 e^{-\mu|a-b|} \text{TV}(f) \quad (4.53)$$

with finite constants $C_7, \mu > 0$ depending only on s and I . Now, the claim follows upon inserting (4.53) into (4.46) and observing (4.35) and (4.52). \square

PROOF OF THEOREM 3.6. As in the proof of Corollary 3.4 we assume without loss of generality that $p \leq 1$, $q \geq 1$ and we define $k := q/p \in \mathbb{N}$. Let $f \in \mathcal{F}_I$. The argument leading to (4.41) in the proof of Corollary 3.4 implies in the present

context the estimate

$$\mathbb{E} [\|T_{f,L}\|_p^q] \leq \sum_{\substack{a_1, \dots, a_k \in \mathbb{Z}^d \\ b_1, \dots, b_k \in \mathbb{Z}^d}} \prod_{l=1}^k \mathbb{E} [\|\chi_{a_l} T_{f,L} \chi_{b_l}\|_p^{pk}]^{1/k}, \quad (4.54)$$

where $T_{f,L} := (f(H_L) - f(H_L^\tau)) - (f(H) - f(H^\tau))$. Below, we will decompose $T_{f,L} = T_{f,L}^{(1)} - T_{f,L}^{(2)}$ in two different ways. The first way is with the terms

$$T_{f,L}^{(1)} := f(H_L) - f(H) \quad \text{and} \quad T_{f,L}^{(2)} := f(H_L^\tau) - f(H^\tau), \quad (4.55)$$

and the second is with the terms

$$T_{f,L}^{(1)} := f(H_L) - f(H_L^\tau) \quad \text{and} \quad T_{f,L}^{(2)} := f(H) - f(H^\tau). \quad (4.56)$$

In any case, the adapted triangle inequality (3.2) and Minkowski's inequality on $L^k(\Omega, \mathbb{P})$ imply

$$\mathbb{E} [\|\chi_{a_l} T_{f,L} \chi_{b_l}\|_p^{pk}]^{1/k} \leq \sum_{j=1}^2 \mathbb{E} [\|\chi_{a_l} T_{f,L}^{(j)} \chi_{b_l}\|_p^{pk}]^{1/k}. \quad (4.57)$$

Moreover, for either choice of the decomposition, the deterministic a priori bound from Lemma B.2 implies $\|\chi_{a_l} T_{f,L}^{(j)} \chi_{b_l}\|_p^{pk-1} \leq C_1 \text{TV}(f)^{pk-1}$ with a finite constant $C_1 = C_{1,p,I} > 0$ that does not depend on ω but only on the lower bound E_0 from (4.17) for the spectra. We conclude that

$$\mathbb{E} [\|T_{f,L}\|_p^q] \leq C_1 \text{TV}(f)^{pk-1} \sum_{\substack{a_1, \dots, a_k \in \mathbb{Z}^d \\ b_1, \dots, b_k \in \mathbb{Z}^d}} \prod_{l=1}^k \left(\sum_{j=1}^2 \mathbb{E} [\|\chi_{a_l} T_{f,L}^{(j)} \chi_{b_l}\|_p^{pk}]^{1/k} \right). \quad (4.58)$$

Next we split the summation over each pair $(a_l, b_l) \in \mathbb{Z}^d \times \mathbb{Z}^d$ into two parts: the box $\Lambda_{L/2}^2 := (\Lambda_{L/2} \times \Lambda_{L/2}) \cap (\mathbb{Z}^d \times \mathbb{Z}^d)$ and its complement $\Lambda_{L/2}^{2,c} := (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus \Lambda_{L/2}^2$. For $(a_l, b_l) \in \Lambda_{L/2}^2$ we compare infinite- and finite-volume operators, that is, we choose the decomposition (4.55). Theorem 3.2 then provides finite constants $C_2 = C_{2,p,I} > 0$ and $\mu_2 = \mu_{2,p,I} > 0$ such that

$$\begin{aligned} & \sum_{(a,b) \in \Lambda_{L/2}^2} \sum_{j=1}^2 \mathbb{E} [\|\chi_{a_l} T_{f,L}^{(j)} \chi_{b_l}\|_p^{pk}]^{1/k} \\ & \leq C_2 \text{TV}(f)^{1/k} \sum_{(a,b) \in \Lambda_{L/2}^2} e^{-\mu_2[(\text{dist}(a, \partial \Lambda_L) + \text{dist}(b, \partial \Lambda_L))/k]} \\ & \leq C_2 \text{TV}(f)^{1/k} (L+1)^{2d} e^{-\mu_2 L/(2k)}. \end{aligned} \quad (4.59)$$

For $(a_l, b_l) \in \Lambda_{L/2}^{2,c}$ we compare unperturbed and perturbed operators, that is, we choose the decomposition (4.56). We fix $0 < s < 1$. Theorem 3.1 then provides

finite constants $C_3 = C_{3,p,s,I} > 0$ and $\mu_3 = \mu_{3,p,s,I} > 0$ such that

$$\begin{aligned} \sum_{(a,b) \in \Lambda_{L/2}^{2,c}} \sum_{j=1}^2 \mathbb{E}[\|\chi_{a_l} T_{f,L}^{(j)} \chi_{b_l}\|_p]^{1/k} &\leq C_3 \text{TV}(f)^{1/k} \sum_{(a,b) \in \Lambda_{L/2}^{2,c}} e^{-\mu_3(|a|+|b|)/k} \\ &\leq 2C_3 \text{TV}(f)^{1/k} \sum_{\substack{a \in \mathbb{Z}^d \\ b \in \mathbb{Z}^d \setminus \Lambda_{L/2}}} e^{-\mu_3(|a|+|b|)/k} \\ &\leq C'_3 \text{TV}(f)^{1/k} (L+1)^{d-1} e^{-\mu_3 L/(4k)}, \end{aligned} \quad (4.60)$$

where $C'_3 = C'_{3,p,s,I} > 0$ is another finite constant. We conclude from (4.58), (4.59) and (4.60) that

$$\mathbb{E}[\|T_{f,L}\|_p^q] \leq C \text{TV}(f)^{pk} e^{-\mu L} \quad (4.61)$$

with finite constants $C = C_{p,q,s,I} > 0$ and $\mu = \mu_{p,q,s,I} > 0$. This proves (3.8).

The almost-sure convergence (3.9) for a super-logarithmically growing sequence of lengths follows from (3.8) with Remark 3.7. \square

5. Proofs of the results from Section 3.2

5.1. Proof of Theorem 3.10.

PROOF OF THEOREM 3.10. We fix $\tau \in [0, 1]$. Let $I \subset \Sigma_{\text{FMB}}$ be a compact interval and $E \in I$. Theorem 3.10(ii) follows from Theorem 3.10(i) and Theorem 3.6 applied to the Fermi function $f = \mathbb{1}_{(-\infty, E]}$.

It remains to prove the left equality in Theorem 3.10(i), because the right equality follows already from Remark 3.11(i). Let $I \subset \Sigma_{\text{FMB}}$ be again a compact interval. Then, Corollary 3.4 and Fubini's Theorem imply

$$\mathbb{E}\left[\int_I dE \|T(E, H, H^\tau)\|_1\right] < \infty. \quad (5.1)$$

Hence, we have $\|T(\cdot, H, H^\tau)\|_1 \in L^1(I)$ almost surely. Thus, the left inequality in Theorem 3.10(i) follows from Lemma 5.1 below, and the proof is complete. \square

It remains to prove the following deterministic lemma.

Lemma 5.1. *Let A and B be two self-adjoint operators in a Hilbert space \mathcal{H} which are bounded from below. We assume that $e^{-A} - e^{-B} \in \mathcal{S}^1$ and that, for some open interval $I \subset \mathbb{R}$, the mapping*

$$I \ni E \mapsto \|T(E, A, B)\|_1 \quad (5.2)$$

is an $L^1(I)$ -function. Then the spectral shift function and the trace of the shift operator coincide, i.e.

$$\xi(E, A, B) = \text{tr}(T(E, A, B)) \quad \text{for a.e. } E \in I. \quad (5.3)$$

Remark 5.2. The assumption $e^{-A} - e^{-B} \in \mathcal{S}^1$ in the lemma is only needed for the spectral shift function to be well defined according to (3.14) and could be relaxed.

PROOF OF LEMMA 5.1. We show that the function $E \mapsto \text{tr}(T(E, A, B))$ satisfies (3.14) for every $f \in C^\infty(\mathbb{R})$ with $\text{supp}(f') \subseteq I$ and $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$.

Let $f \in C^\infty(\mathbb{R})$ be such a function. Assumption (5.2) implies that

$$\mathbb{R} \ni E \mapsto |f'(E)| \|T(E, A, B)\|_1 \in L^1(\mathbb{R}). \quad (5.4)$$

Hence, $\int_{\mathbb{R}} dE f'(E) T(E, A, B)$ is a trace-norm convergent Bochner integral. Moreover, the identity

$$f(B) - f(A) = \int_{\mathbb{R}} dE f'(E) T(E, A, B) \quad (5.5)$$

holds, as we argue below. Since the mapping $\mathcal{S}^1 \ni B \mapsto \text{tr}(B)$ is a bounded linear functional on \mathcal{S}^1 , it interchanges with the Bochner integral, and (5.5) implies the assertion

$$\text{tr}(f(B) - f(A)) = \int_{\mathbb{R}} dE f'(E) \text{tr}(T(E, A, B)). \quad (5.6)$$

It remains to prove (5.5). We fix $\phi \in \mathcal{H}$ and compute, using Fubini's theorem,

$$\langle \phi, f(A)\phi \rangle = - \int_{\mathbb{R}} d\mu_{\phi, \phi}^A(\lambda) \int_{\lambda}^{\infty} dE f'(E) = - \int_{\mathbb{R}} dE f'(E) \langle \phi, \mathbf{1}_{(-\infty, E]}(A)\phi \rangle, \quad (5.7)$$

where we introduced the spectral measure $\mu_{\phi, \phi}^A := \langle \phi, \mathbf{1}_{\cdot}(A)\phi \rangle$. The analogous computation for B gives

$$\begin{aligned} \langle \phi, (f(B) - f(A))\phi \rangle &= \int_{\mathbb{R}} dE f'(E) \langle \phi, T(E, A, B)\phi \rangle \\ &= \langle \phi, \left(\int_{\mathbb{R}} dE f'(E) T(E, A, B) \right) \phi \rangle, \end{aligned} \quad (5.8)$$

where the last equality follows again from the continuity of the functional $\mathcal{S}^1 \ni B \mapsto \langle \phi, B\phi \rangle$, which therefore interchanges with the Bochner integral. The operator equality (5.5) follows by polarisation. \square

5.2. Proof of Lemma 3.14 and Theorem 3.16.

PROOF OF LEMMA 3.14. We fix $E, E' \in I$ arbitrary with $E < E'$ and define $J := [E, E']$. The estimate (4.32) with $f = h = \mathbf{1}_J$ implies

$$\begin{aligned} &\mathbb{E}[\|T(E, H_{(L)}, H_{(L)}^\tau) - T(E', H_{(L)}, H_{(L)}^\tau)\|_1] \\ &\leq \mathbb{E}[\|\mathbf{1}_J(H_{(L)}) - \mathbf{1}_J(H_{(L)}^\tau)\|_1] \\ &\leq C_\theta \sum_{a, b \in \mathbb{Z}^d} \mathbb{E}[\|\chi_a(\mathbf{1}_J(H_{(L)}) - \mathbf{1}_J(H_{(L)}^\tau))\chi_b\|]^\theta, \end{aligned} \quad (5.9)$$

where $\theta \in (0, 1)$ and the constant C_θ is independent of L and $\tau \in [0, 1]$. We will apply the Helffer-Sjöstrand formula to the operator $\mathbf{1}_J(H_{(L)}^{(\tau)})$ and need an appropriate cutoff function for this purpose. Let $g \in C_c^\infty(\mathbb{R})$ be a smooth indicator function such that $g(x) := 1$ for $x \in [-1, 1]$, $g(x) := 0$ for $x \in \mathbb{R} \setminus [-2, 2]$, $\|g\|_\infty \leq 1$ and $\|g'\|_\infty \leq 2$. We define the centre $J_c := (E + E')/2$ of the interval J and the cutoff function $\Xi_J \in C_c^\infty(\mathbb{R}^2)$ by

$$\Xi_J(x, y) = g\left(\frac{x - J_c}{|J|}\right) g\left(\frac{y}{|J|}\right), \quad (x, y) \in \mathbb{R}^2. \quad (5.10)$$

Let $\zeta_{\mathbb{1}_J}$ be the complex measure defined in (4.4) with the cutoff function (5.10). We note that

$$\text{supp}(\zeta_{\mathbb{1}_J}) \subseteq J \times [-2|J|, 2|J|]. \quad (5.11)$$

We now apply Lemma 4.1 to the difference in (5.9) and obtain for all $a, b \in \mathbb{Z}^d$

$$\mathbb{E} \left[\left\| \chi_a (\mathbb{1}_J(H_{(L)}) - \mathbb{1}_J(H_{(L)}^\tau)) \chi_b \right\| \right] \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} d|\zeta_{\mathbb{1}_J}|(x, y) \mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\| \right], \quad (5.12)$$

where we abbreviated $z := x + iy$, $R_z := R_z(H_{(L)})$ and $R_z^\tau := R_z(H_{(L)}^\tau)$. Let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ and let $\delta \in (0, 1/p)$. Then, we estimate with the help of the resolvent equation

$$\begin{aligned} & \mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\| \right] \\ &= \mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\| \right]^{1/p} \mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\| \right]^{1/q} \\ &\leq \frac{2^{\delta/p} (\tau \|W\|_\infty)^{1/q}}{|y|^{\delta/p + (1+\delta)/q}} \mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\|^{1-\delta} \right]^{1/p} \\ &\quad \times \sum_{c \in \Gamma_W} \mathbb{E} \left[\left\| \chi_a R_z \chi_c \right\|^{(1-\delta)/2} \left\| \chi_c R_z^\tau \chi_b \right\|^{(1-\delta)/2} \right]^{1/q} \\ &\leq \frac{2^{\delta/p} (\tau \|W\|_\infty)^{1/q}}{|y|^{\delta+1/q}} \left(\mathbb{E} \left[\left\| \chi_a R_z \chi_b \right\|^{1-\delta} \right] + \mathbb{E} \left[\left\| \chi_a R_z^\tau \chi_b \right\|^{1-\delta} \right] \right)^{1/p} \\ &\quad \times \sum_{c \in \Gamma_W} \left(\mathbb{E} \left[\left\| \chi_a R_z \chi_c \right\|^{1-\delta} \right] \mathbb{E} \left[\left\| \chi_c R_z^\tau \chi_b \right\|^{1-\delta} \right] \right)^{1/(2q)}, \end{aligned} \quad (5.13)$$

where Γ_W was defined in (4.20). For $(x, y) \in \text{supp}(\zeta_{\mathbb{1}_J})$ we have $x \in J \subseteq I \subset \Sigma_{\text{FMB}}$ so that each expectation can be bounded with the localisation estimate from Lemma A.1, which is uniform in $\tau \in [0, 1]$, $L > 0$ and $J \subseteq I$. Treating the resulting c -sum of exponentials as in (4.28) and throwing away the decay in $|a - b|$, we arrive at

$$\mathbb{E} \left[\left\| \chi_a (R_z - R_z^\tau) \chi_b \right\| \right] \leq C_1 \frac{e^{-\mu_1(|a|+|b|)}}{|y|^{\delta+1/q}} \quad (5.14)$$

with finite constants $C_1, \mu_1 > 0$ that depend only on p, δ and I . This estimate needs to be inserted into (5.12), so the following integral is relevant

$$\int_{\mathbb{R}^2} \frac{d|\zeta_{\mathbb{1}_J}|(x, y)}{|y|^{\delta+1/q}} \leq C_2 |J|^{p^{-1}-\delta}. \quad (5.15)$$

To obtain the bound, we used $\delta < 1/p$ and applied (4.10) with the particular choice (5.10) for the cutoff function Ξ_J . The constant $C_2 > 0$ depends only on p, δ and I . Collecting the estimates (5.9), (5.12), (5.14) and (5.15), we conclude that

$$\mathbb{E} \left[\left\| T(E, H_{(L)}, H_{(L)}^\tau) - T(E', H_{(L)}, H_{(L)}^\tau) \right\|_1 \right] \leq C_3 |J|^{\theta(p^{-1}-\delta)} \quad (5.16)$$

with a finite constant $C_3 > 0$ that depends on θ, p, δ and I , but not on $J \subseteq I$, $L > 0$ or $\tau \in [0, 1]$. The lemma follows, because the exponent $\theta(p^{-1} - \delta)$ may take on any value in $(0, 1)$. \square

Our proof of Theorem 3.16 uses the scale-free unique continuation principle of [Kle13, NTTV15] applied to averaged local traces of non-negative functions of *infinite-volume* ergodic random Schrödinger operators.

Lemma 5.3. *Let $E \in \mathbb{R}$ and let $\Gamma \subset \mathbb{R}^d$ be a Borel set with $\text{int}(\Gamma) \neq \emptyset$. Then there exists a finite constant $\gamma > 0$ such that for every non-negative, measurable function $f : \mathbb{R} \rightarrow [0, \infty)$ with support $\text{supp}(f) \subseteq (-\infty, E]$ the lower bound*

$$\mathbb{E}[\text{tr}(\mathbb{1}_\Gamma f(H))] \geq \gamma \int_{\mathbb{R}} dE' \mathcal{N}'(E') f(E') \quad (5.17)$$

holds. If the operator $\mathbb{1}_\Gamma f(H)$ is not trace class, we define its trace to be $+\infty$.

PROOF. Even though a unique continuation principle for infinite-volume operators is known [TV16], it seems more convenient here to use the one for finite-volume operators cited above. By usual integration theory, the lemma follows if it is proven for indicator functions f of arbitrary Borel sets $B \subseteq (-\infty, E]$. In turn, by the comparison theorem for measures, see e.g. [Els05, Thm. II.5.8], it is enough to prove it for semi-open intervals $I \subset (-\infty, E]$.

So let I be such an interval. Since the probability for the endpoints of I to coincide with an eigenvalue of H is zero, we obtain from the definition and self-averaging of the integrated density of states

$$\int_I dE' \mathcal{N}'(E') = \lim_{L \rightarrow \infty} \frac{1}{L^d} \mathbb{E}[\text{tr}(\mathbb{1}_I(H_L))]. \quad (5.18)$$

The left-hand side of (5.17) is monotone in Γ and invariant under \mathbb{Z}^d -translations of Γ . Therefore we assume without loss of generality that there exists an interior point of Γ inside the unit cube Λ_1 and that $\Gamma \subseteq \Lambda_1$. Then, the scale-free unique continuation principle for spectral projections, see [NTTV15, Cor.] or [Kle13, Thm. 1.1], provides us with the deterministic estimate

$$\mathbb{1}_I(H_L) \leq \frac{1}{\gamma} \mathbb{1}_I(H_L) \mathbb{1}_{\Gamma_L} \mathbb{1}_I(H_L) \quad (5.19)$$

for $L \in \mathbb{N}$, where $\Gamma_L := \Lambda_L \cap (\bigcup_{k \in \mathbb{Z}^d} (k + \Gamma))$ and the finite non-random constant $\gamma > 0$ depends only on d , Γ , E and

$$\mathcal{V} := \|V_0\|_\infty + |\lambda| \left\| \sum_{k \in \mathbb{Z}^d} u_k \right\|_\infty < \infty, \quad (5.20)$$

see Assumptions (K) and (V2). Next, we insert (5.19) into (5.18), exploit cyclicity of the trace and add and subtract the desired L -independent term. This yields

$$\begin{aligned} \int_I dE' \mathcal{N}'(E') &\leq \frac{1}{\gamma} \mathbb{E}[\text{tr}(\mathbb{1}_\Gamma \mathbb{1}_I(H))] \\ &+ \frac{1}{\gamma} \liminf_{L \rightarrow \infty} \left(\frac{1}{L^d} \mathbb{E}[\text{tr}(\mathbb{1}_{\Gamma_L} \mathbb{1}_I(H_L))] - \mathbb{E}[\text{tr}(\mathbb{1}_\Gamma \mathbb{1}_I(H))] \right). \end{aligned} \quad (5.21)$$

It remains to show that the error term in the second line of (5.21) vanishes as $L \rightarrow \infty$. This is a question of convergence of measures. Since a given real number is not an eigenvalue of H with probability one, the limiting measure has no atoms, and the notion of vague convergence is sufficient. Thus, we consider the Laplace transforms, and the lemma follows if

$$\lim_{\substack{L \rightarrow \infty \\ L \text{ odd}}} \frac{1}{L^d} \mathbb{E}[\text{tr}(\mathbb{1}_{\Gamma_L} (e^{-tH_L} - e^{-tH}))] = 0 \quad (5.22)$$

holds for every $t > 0$ [Fel71, Thm. 2a in Sect. XIII.1]. Here, we reformulated the term with the \mathbb{Z}^d -ergodic infinite-volume Hamiltonian by using the covariance relation for H and the disjoint decomposition $\Gamma_L = \bigcup_{k \in \Lambda_L \cap \mathbb{Z}^d} (k + \Gamma)$, which holds for L odd. We analyse the semigroups in the Feynman-Kac representation [Sim05a], see also [BHL00, Sect. 6], and obtain

$$\begin{aligned} & \left| \frac{1}{L^d} \mathbb{E} [\operatorname{tr} (\mathbb{1}_{\Gamma_L} (e^{-tH_L} - e^{-tH}))] \right| \\ & \leq \frac{e^{t\nu}}{(2\pi t)^{d/2}} \int_{\Gamma_L} \frac{dx}{L^d} \mathbf{E}_{0,x}^{t,x} (1 - \chi_{\Lambda_L}^t) \leq \frac{e^{t\nu}}{(2\pi t)^{d/2}} \int_{\Lambda_L} \frac{dx}{L^d} \mathbf{E}_{0,x}^{t,x} (1 - \chi_{\Lambda_L}^t), \end{aligned} \quad (5.23)$$

where $\mathbf{E}_{0,x}^{t,x}$ denotes the Brownian-bridge expectation over continuous closed paths starting at time 0 at x and returning to x at time t , and $\chi_{\Lambda_L}^t$ is the indicator function of the set of Brownian-bridge paths which stay inside Λ_L for all times up to t . But the integral on the right-hand side of (5.23) vanishes as $L \rightarrow \infty$ according to [Kir89, p. 341]. \square

We are now ready for the

PROOF OF THEOREM 3.16. Throughout the proof we abbreviate the spectral shift functions by $\xi_{(L)} := \xi(\cdot, H_{(L)}, H_{(L)}^\tau)$ and choose representatives which coincide with the trace of the corresponding spectral shift operator on Σ_{FMB} . Since $\xi_{(L)}$ depends monotonously on the perturbation and $W \geq C u_0$, we assume without loss of generality that $W = C u_0$.

By Lemma 3.14, the function $E \mapsto \mathbb{E}[\xi(E)]$ is Hölder continuous on compact intervals in Σ_{FMB} for any $\alpha \in (0, 1)$. Let $E \in \Sigma_{\text{FMB}}$ and $\varepsilon_0 > 0$ such that $I_{\varepsilon_0} := [E - \varepsilon_0, E + \varepsilon_0] \subset \Sigma_{\text{FMB}}$. Consider any $\varepsilon \in (0, \varepsilon_0]$. Then

$$\mathbb{E}[\xi(E)] \geq \frac{1}{2\varepsilon} \int_{I_\varepsilon} dE' \mathbb{E}[\xi(E')] - C_1 \varepsilon^\alpha, \quad (5.24)$$

where the Hölder constant depends on α , E and ε_0 , but not on ε . We denote by $\mathbb{E}_{\neq 0}[\cdot]$ the averaging with respect to all random variables but ω_0 and infer from the Birman-Solomyak formula [BS75]

$$\begin{aligned} \int_{I_\varepsilon} dE' \mathbb{E}[\xi(E')] &= \int_0^1 ds \mathbb{E}[\operatorname{tr}(C u_0 \mathbb{1}_{I_\varepsilon}(H + s C u_0))] \\ &= \int_0^C ds \int_0^1 d\omega_0 \rho(\omega_0) \mathbb{E}_{\neq 0}[\operatorname{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H + s u_0))]. \end{aligned} \quad (5.25)$$

We fix a parameter $s_0 \in (0, \min\{1, C\}]$ to be determined later. Performing the change of variables $\omega_0 \mapsto \omega_0 + s$ and restricting first the s -integration to $[0, s_0]$ and then the ω_0 -integration to $[s_0, 1]$, we obtain the estimate

$$\begin{aligned} \int_{I_\varepsilon} dE' \mathbb{E}[\xi(E')] &\geq s_0 \rho_- \int_{s_0}^1 d\omega_0 \mathbb{E}_{\neq 0}[\operatorname{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H))] \\ &\geq s_0 \rho_- \left(\frac{1}{\rho_+} \mathbb{E}[\operatorname{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H))] - \int_0^{s_0} d\omega_0 \mathbb{E}_{\neq 0}[\operatorname{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H))] \right) \\ &\geq s_0 \rho_- \left(\frac{\gamma \mathcal{N}(I_\varepsilon)}{\rho_+} - J \right). \end{aligned} \quad (5.26)$$

Here, the last inequality follows from the right estimate in (3.28) together with the unique continuation principle stated in Lemma 5.3. Furthermore, we introduced $\rho_+ := \text{ess sup}_{x \in [0,1]} \rho(x) < \infty$ and

$$J := \int_0^{s_0} d\omega_0 \mathbb{E}_{\neq 0} [\text{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H))] . \quad (5.27)$$

To estimate J we first exchange the operator H by its finite-volume restriction H_L . The error arising from this modification of the operator can be bounded by Theorem 3.2, which yields finite constants $C_2, \mu > 0$ such that

$$\mathbb{E} [|\text{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H)) - \text{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H_L))|] \leq C_2 e^{-\mu L} \quad (5.28)$$

for every $L > 0$. The estimate (5.28) implies the bound

$$J \leq s_0 \sup_{\omega_0 \in [0, s_0]} \mathbb{E}_{\neq 0} [\text{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H_L))] + \frac{C_2}{\rho_-} e^{-\mu L} \quad (5.29)$$

for every $L > 0$. Let $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^d$ be the unit vector along the first coordinate axis. We define the subset of lattice points $A_L := (e_1 + (3\mathbb{Z})^d) \cap \Lambda_L$ and introduce the random background operator

$$\tilde{H}_L := H_{0,L} + \lambda \sum_{k \in \mathbb{Z}^d \setminus A_L} \omega_k u_k, \quad (5.30)$$

where $H_{0,L}$ is the Dirichlet restriction of H_0 to Λ_L . However, for any fixed realisation of coupling constants $\{\omega_k\}_{k \in \mathbb{Z}^d \setminus A_L}$, we view \tilde{H}_L as a non-random operator with a non-periodic background potential that is bounded uniformly in $\{\omega_k\}_{k \in \mathbb{Z}^d \setminus A_L}$. Now, the Dirichlet restriction H_L takes the form

$$H_L = \tilde{H}_L + \lambda \sum_{k \in A_L} \omega_k u_k \quad (5.31)$$

and, after scaling by a factor $1/3$ and (if required) introducing a energy shift, it constitutes a *crooked Anderson Hamiltonian* in the sense of [Kle13]. We apply the Wegner estimate [Kle13, Thm. 1.4] and obtain a finite constant $C_3 > 0$ such that

$$\begin{aligned} \sup_{\omega_k \in [0,1], k \in \mathbb{Z}^d \setminus A_L} \mathbb{E}_{A_L} [\text{tr}(u_0 \mathbb{1}_{I_\varepsilon}(H_L))] &\leq \|u_0\|_\infty \sup_{\omega_k \in [0,1], k \in \mathbb{Z}^d \setminus A_L} \mathbb{E}_{A_L} [\text{tr}(\mathbb{1}_{I_\varepsilon}(H_L))] \\ &\leq C_3 2\varepsilon L^d, \end{aligned} \quad (5.32)$$

where $\mathbb{E}_{A_L}[\cdot]$ denotes the average over the couplings $\{\omega_k\}_{k \in A_L}$. The estimate (5.32) holds for all $\varepsilon \in (0, \varepsilon_1]$ and all length-scales $L \geq L_1$ such that $L/3 \in \mathbb{N}$ is odd, and ε_1 and L_1 depend only on model parameters. We insert (5.32) into (5.29) and obtain

$$J \leq C_3 2\varepsilon s_0 L^d + \frac{C_2}{\rho_-} e^{-\mu L}. \quad (5.33)$$

Combining (5.24), (5.26) and (5.33), we conclude

$$\mathbb{E} [\xi(E)] \geq \rho_- s_0 \left(\frac{\gamma}{\rho_+} \frac{\mathcal{N}(I_\varepsilon)}{2\varepsilon} - C_3 s_0 L^d - \frac{C_2}{\rho_-} \frac{e^{-\mu L}}{2\varepsilon} \right) - C_1 \varepsilon^\alpha \quad (5.34)$$

for all $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\}]$, all $s_0 \in (0, \min\{1, C\}]$ and all $L \geq L_1$ such that $L/3 \in \mathbb{N}$ is odd.

Now, suppose that E is a Lebesgue point of the integrated density of states \mathcal{N} , which is the case Lebesgue-almost everywhere. Then we have $\mathcal{N}(I_\varepsilon)/2\varepsilon \rightarrow \mathcal{N}'(E)$

as $\varepsilon \downarrow 0$. Suppose also that $\mathcal{N}'(E) > 0$. Then, there exists $\varepsilon_2 \in (0, \min\{\varepsilon_0, \varepsilon_1\}]$ such that $\mathcal{N}(I_\varepsilon)/2\varepsilon \geq \mathcal{N}'(E)/2$ for every $\varepsilon \in (0, \varepsilon_2]$. Finally, we choose $L \geq L_1$ with $L/3 \in \mathbb{N}$ odd so large that $\varepsilon := L^{-4d/\alpha} \leq \varepsilon_2$ and $s_0 := L^{-2d} \leq \min\{1, C\}$. In this case (5.34) yields

$$\mathbb{E}[\xi(E)] \geq \frac{\gamma\rho_-}{2\rho_+} L^{-2d} \left(\mathcal{N}'(E) - O(L^{-d}) \right). \quad (5.35)$$

The right-hand side of (5.35) is strictly positive by possibly enlarging L even further. \square

6. Proofs of the results from Section 3.3

PROOF OF LEMMA 3.18. Consider the $N \times N$ -matrix $M := (\langle \varphi_j^L, \psi_k^L \rangle)_{1 \leq j, k \leq N}$. Then, the matrix entries of MM^* and M^*M read

$$\begin{aligned} (MM^*)_{jl} &= \sum_{k=1}^N \langle \varphi_j^L, \psi_k^L \rangle \langle \psi_k^L, \varphi_l^L \rangle = \langle \varphi_j^L, PQP\varphi_l^L \rangle, \\ (M^*M)_{jl} &= \sum_{k=1}^N \langle \psi_j^L, \varphi_k^L \rangle \langle \varphi_k^L, \psi_l^L \rangle = \langle \psi_j^L, QPQ\psi_l^L \rangle \end{aligned} \quad (6.1)$$

for $1 \leq j, l \leq N$. Thus, $S_{N,L}$ can be written as

$$S_{N,L}^2 = \det(MM^*) = \det(PQP|_{\text{ran } P}) = \det(\mathbb{1} - P(\mathbb{1} - Q)P), \quad (6.2)$$

and likewise as

$$\begin{aligned} S_{N,L}^2 &= \det(M^*M) = \det(QPQ|_{\text{ran } Q}) = \det(\mathbb{1} - Q(\mathbb{1} - P)Q) \\ &= \det(\mathbb{1} - (\mathbb{1} - P)Q(\mathbb{1} - P)). \end{aligned} \quad (6.3)$$

Here, the last equality follows from the fact that the non-zero singular values of $Q(\mathbb{1} - P)$ coincide with the non-zero singular values of its adjoint $(\mathbb{1} - P)Q$. The determinants in (6.2) and (6.3) are well-defined Fredholm determinants because $P(\mathbb{1} - Q)P$ and $Q(\mathbb{1} - P)Q$ are of finite rank. Multiplying the expressions in (6.2) and (6.3) yields

$$\begin{aligned} S_{N,L}^4 &= \det(\mathbb{1} - P(\mathbb{1} - Q)P) \det(\mathbb{1} - (\mathbb{1} - P)Q(\mathbb{1} - P)) \\ &= \det(\mathbb{1} - P(\mathbb{1} - Q)P - (\mathbb{1} - P)Q(\mathbb{1} - P)). \end{aligned} \quad (6.4)$$

Now, Lemma C.1 completes the proof of the lemma. \square

The remainder of this section is devoted to the proof of Theorem 3.19. For $L > 0$ and $E \in \mathbb{R}$ we abbreviate the index and the shift operator of $H_{(L)}$ and $H_{(L)}^\tau$ by

$$\theta_{(L)}(E) := \theta(E, H_{(L)}, H_{(L)}^\tau) \quad \text{and} \quad T_{(L)}(E) := T(E, H_{(L)}, H_{(L)}^\tau), \quad (6.5)$$

respectively.

PROOF OF THEOREM 3.19. We address the different parts of the theorem in the order (ii), (iii), (iv), (i).

PART (ii). Let $E \in \Sigma_{\text{FMB}}$. Then Corollary 3.4 implies that $T(E) \in \mathcal{S}^2$ almost surely, and we have $S(E) = \det(\mathbb{1} - T(E)^2)^{1/4}$ almost surely. Clearly, if 1 is an eigenvalue of $T(E)^2$, then $S(E) = 0$. Conversely, suppose that 1 is not an eigenvalue

of $T(E)^2$. Observing (3.13), we denote by $1 > \|T(E)\|^2 = b_1 \geq b_2 \geq \dots \geq 0$ the non-increasingly ordered sequence of eigenvalues of $T(E)^2$. Then we obtain

$$\begin{aligned} S(E)^4 &= \exp \left\{ \sum_{n \in \mathbb{N}} \ln(1 - b_n) \right\} = \exp \left\{ - \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{b_n^k}{k} \right\} \\ &\geq \exp \left\{ - \sum_{n \in \mathbb{N}} b_n \sum_{k \in \mathbb{N}} b_1^{k-1} \right\} = \exp \left\{ - \frac{\|T(E)\|_2^2}{1 - \|T(E)\|^2} \right\} > 0. \end{aligned} \quad (6.6)$$

PART (iii). We fix a compact interval $I \subset \Sigma_{\text{FMB}}$, $E \in I$, a constant $c > 0$ and define the event $\Omega_0 := \{\|T(E)\|_1 < 1\}$. Markov's inequality implies

$$\begin{aligned} \mathbb{E}[S(E)] &\geq e^{-c/4} \mathbb{P}[S(E)^4 \geq e^{-c}] \geq e^{-c/4} \mathbb{P}[\Omega_0 \cap \{S(E)^4 \geq e^{-c}\}] \\ &\geq e^{-c/4} \mathbb{P}\left[\Omega_0 \cap \left\{ \exp\left(-\frac{\|T(E)\|_2^2}{1 - \|T(E)\|^2}\right) \geq e^{-c} \right\}\right], \end{aligned} \quad (6.7)$$

where we used (6.6) to obtain the last inequality. Since $\|T(E)\| \leq \|T(E)\|_1 < 1$ on Ω_0 and, thus, $\|T(E)\|_2^2 \leq \|T(E)\| \|T(E)\|_1 < \|T(E)\|_1$, we conclude that the fraction inside the exponent in the last line of (6.7) is bounded from above by $\|T(E)\|_1 / (1 - \|T(E)\|_1)$. Therefore we get

$$\mathbb{E}[S(E)] \geq e^{-c/4} \mathbb{P}\left[\|T(E)\|_1 \leq \frac{c}{c+1}\right]. \quad (6.8)$$

We apply Markov's inequality again and obtain

$$\mathbb{P}\left[\|T(E)\|_1 \leq \frac{c}{c+1}\right] = 1 - \mathbb{P}\left[\|T(E)\|_1 > \frac{c}{c+1}\right] \geq 1 - \frac{c+1}{c} \mathbb{E}[\|T(E)\|_1] \quad (6.9)$$

so that

$$\mathbb{E}[S(E)] \geq e^{-c/4} \left(1 - \frac{c+1}{c} \mathbb{E}[\|T(E)\|_1]\right). \quad (6.10)$$

Corollary 3.4 gives $\sup_{E \in I} \mathbb{E}[\|T(E)\|_1] \rightarrow 0$ as $\tau \rightarrow 0$. Hence, the claim follows by letting $c \downarrow 0$.

PART (iv). We restrict ourselves to $W \geq 0$, the proof for $W \leq 0$ follows along the same lines. Since $\theta(E, H, H^\tau) = \xi(E, H, H^\tau)$ for a.e. $E \in \Sigma_{\text{FMB}}$ almost surely by Theorem 3.10(i), the equivalence (3.50) implies

$$\mathbb{P}[S(E) = 0] = \mathbb{P}[\theta(E, H, H^\tau) > 0] = \mathbb{P}[\xi(E, H, H^\tau) > 0] \quad (6.11)$$

for a.e. $E \in \Sigma_{\text{FMB}}$. Hence, the claim follows from Theorem 3.16.

PART (i). We fix $E \in \Sigma_{\text{FMB}} \cap \text{int}(\Sigma)$. The almost-sure statement (3.44) implies (3.43) by dominated convergence and a subsubsequence argument applied to the \mathbb{R} -valued sequence $(\mathbb{E}[S_L(E)])_{L>0}$. Thus it remains to prove (3.44).

Let $(L_n)_{n \in \mathbb{N}}$ be a sequence with $L_n / \ln n \rightarrow \infty$ as $n \rightarrow \infty$. Since the quantities $\theta_{(L_n)}(E)$ are all integer-valued, (3.22) implies that there is a random variable $n_0 := n_0(E) : \Omega \rightarrow \mathbb{N}$ such that almost surely

$$\theta(E) = \theta_{L_n}(E) \quad \text{for all } n \geq n_0. \quad (6.12)$$

Case 1: $\theta(E) = 0$. Let $n \geq n_0$. Because of (6.12), (3.20) and (3.21) for $L = L_n$ we have the representation (3.41) for the finite-volume overlap in this case. The

continuity of the Fredholm determinant with respect to the trace norm [Sim05b, Thm. 3.4] implies

$$\begin{aligned} |S_{L_n}(E)^4 - S(E)^4| &= |\det(\mathbb{1} - T_{L_n}(E)^2) - \det(\mathbb{1} - T(E)^2)| \\ &\leq 2\|T_{L_n}(E) - T(E)\|_1 \exp(\|T_{L_n}(E)\|_2^2 + \|T(E)\|_2^2 + 1). \end{aligned} \quad (6.13)$$

Theorem 3.6 now yields the almost-sure convergence

$$\lim_{n \rightarrow \infty} \|T_{L_n}(E) - T(E)\|_1 \exp(\|T_{L_n}(E)\|_2^2 + \|T(E)\|_2^2 + 1) = 0. \quad (6.14)$$

This and (6.13) imply the desired convergence (3.44) in the case $\theta(E) = 0$.

Case 2: $\theta(E) \neq 0$. We assume without loss of generality that $\theta(E) > 0$. The other case follows along the same lines. Because of the equivalence (3.50) we have $S(E) = 0$ in this case, which, in turn, is equivalent $1 \in \sigma(T(E)^2)$ by (3.45). We abbreviate $P_n := P_{N_{L_n}(E), L_n}$ and $Q_n := Q_{N_{L_n}(E), L_n}$. Let us assume the strong convergence

$$\text{s-}\lim_{n \rightarrow \infty} (P_n - Q_n)^2 = T(E)^2 \quad \text{almost surely} \quad (6.15)$$

for the time being. Since $1 \in \sigma(T(E)^2)$, it implies almost surely the existence of a sequence of eigenvalues $(\alpha_n)_{n \in \mathbb{N}}$ with $\alpha_n \in \sigma((P_n - Q_n)^2)$ and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$ [RS80, Thm. VIII.24(a)]. Moreover, $0 \leq (P_n - Q_n)^2 \leq 1$, and we conclude (note that $N_{L_n}(E) \neq 0$ for n large enough because $E \in \text{int}(\Sigma)$, whence the first line in (3.40) applies)

$$S_{L_n}(E)^4 = \det(\mathbb{1} - (P_n - Q_n)^2) \leq 1 - \alpha_n \rightarrow 0 \quad (6.16)$$

as $n \rightarrow \infty$ almost surely. This is the assertion in the case $\theta(E) \neq 0$.

Thus, it remains to prove (6.15). For this it suffices to show strong convergence $P_n - Q_n \rightarrow T(E)$ as $n \rightarrow \infty$. Let $\eta \in L^2(\mathbb{R}^d)$. We add and subtract the term $T_{L_n}(E)$ and estimate

$$\|(P_n - Q_n - T(E))\eta\| \leq \|(\mathbb{1}_{(-\infty, E]}(H_{L_n}^\tau) - Q_n)\eta\| + \|(T_{L_n}(E) - T(E))\eta\|, \quad (6.17)$$

where we used that $P_n = \mathbb{1}_{(-\infty, E]}(H_{L_n})$ by definition of the particle number (3.39). The second term on the right-hand side of (6.17) converges to 0 as $n \rightarrow \infty$ almost surely by Theorem 3.6. As to the first term on the right-hand side we recall from (6.12), (3.20) and $\text{tr } P_n = \text{tr } Q_n$ that

$$\theta(E) = \theta_{L_n}(E) = \text{tr}(T_{L_n}(E)) = \text{tr}(Q_n - \mathbb{1}_{(-\infty, E]}(H_{L_n}^\tau)) \quad (6.18)$$

almost surely for $n \geq n_0$. Since we assumed that $\theta(E) > 0$ and Q_n is the orthogonal projection on the eigenspaces of all eigenvalues up to the $N_{L_n}(E)$ th eigenvalue of $H_{L_n}^\tau$, we obtain that

$$\mu_{N_{L_n}(E) - \theta(E)} \leq E \leq \mu_{N_{L_n}(E) - \theta(E) + 1} \leq \mu_{N_{L_n}(E)} \quad (6.19)$$

and that

$$Q_n - \mathbb{1}_{(-\infty, E]}(H_{L_n}^\tau) = \sum_{k=0}^{\theta(E)-1} |\psi_{N_{L_n}(E)-k}^{L_n}\rangle \langle \psi_{N_{L_n}(E)-k}^{L_n}| \quad (6.20)$$

hold almost surely for $n \geq n_0$. Since $E \in \text{int}(\Sigma)$ and $\Sigma = \sigma_{\text{ess}}(H^\tau)$, we obtain for any $\varepsilon > 0$ by Fatou's lemma and strong resolvent convergence of $H_{L_n}^\tau$ to H^τ that

$$\lim_{n \rightarrow \infty} \text{tr}(\mathbb{1}_{[E, E+\varepsilon)}(H_{L_n}^\tau)) \geq \text{tr}(\mathbb{1}_{[E, E+\varepsilon)}(H^\tau)) = \infty \quad (6.21)$$

almost surely. Since $\theta(E)$ is a finite number, this implies that for any $\varepsilon > 0$ there is an $n_1 > 0$ such that

$$\sum_{k=0}^{\theta(E)-1} |\psi_{N_{L_n}(E)-k}^{L_n} \rangle \langle \psi_{N_{L_n}(E)-k}^{L_n}| \leq \mathbb{1}_{[E, E+\varepsilon)}(H^\tau) \quad (6.22)$$

for all $n \geq n_1$ almost surely. This yields for any fixed $\varepsilon > 0$ the bound

$$\limsup_{n \rightarrow \infty} \|(Q_n - \mathbb{1}_{(-\infty, E]}(H_{L_n}^\tau))\eta\| \leq \limsup_{n \rightarrow \infty} \|\mathbb{1}_{[E, E+\varepsilon)}(H_{L_n}^\tau)\eta\| \quad (6.23)$$

almost surely. Finally, we fix a null sequence $(\varepsilon_l)_{l \in \mathbb{N}} \subset (0, \infty)$. Lemma A.4 implies that $E, E + \varepsilon_l$, $l \in \mathbb{N}$, are no eigenvalues of H^τ almost surely. Therefore we have strong convergence $\mathbb{1}_{[E, E+\varepsilon_l)}(H_{L_n}^\tau) \rightarrow \mathbb{1}_{[E, E+\varepsilon_l)}(H^\tau)$ as $L \rightarrow \infty$ almost surely [RS80, Thm. VIII.24(b)]. This and (6.23) lead to

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|(Q_n - \mathbb{1}_{(-\infty, E]}(H_{L_n}^\tau))\eta\| &\leq \limsup_{l \rightarrow \infty} \|\mathbb{1}_{[E, E+\varepsilon_l)}(H^\tau)\eta\| \\ &= \|\mathbb{1}_{\{E\}}(H^\tau)\eta\| = 0 \end{aligned} \quad (6.24)$$

almost surely, where the last equality follows again from E being almost surely not an eigenvalue of H^τ . This proves (6.15). \square

Appendix A. Stability of fractional-moment bounds

Here, we prove that the fractional-moment bounds given in Definition 2.1 are stable under local perturbations. Among others, we show that $\Sigma_{\text{FMB}}(H) = \Sigma_{\text{FMB}}(H+W)$ for perturbations $W \in L_c^\infty(\mathbb{R}^d)$.

Lemma A.1. *Let $I \subset \Sigma_{\text{FMB}}$ be a compact interval. Then for any fixed $0 < s < 1$ there exist constants $C, \mu > 0$ such that for all $G \subseteq \mathbb{R}^d$ open and $a, b \in \mathbb{R}^d$*

$$\sup_{\tau \in [0, 1]} \sup_{E \in I, \eta \neq 0} \mathbb{E} [\|\chi_a R_{E+i\eta}(H_G^\tau) \chi_b\|^s] \leq C e^{-\mu|a-b|} \quad (A.1)$$

holds. In particular, $\Sigma_{\text{FMB}}(H) = \Sigma_{\text{FMB}}(H^\tau)$ for every $\tau \in [0, 1]$.

In order to prove Lemma A.1, we need to know that if fractional-moment estimates hold for one exponent $0 < s < 1$, then they hold for every exponent $0 < s < 1$. This was proven in [ASFH01, Lemma B.2] in the discrete case. This proof directly extends to the continuum.

Lemma A.2. *If, for some $0 < s_0 < 1$, there exist constants $C_{s_0}, \mu_{s_0} > 0$ such that for all $G \subseteq \mathbb{R}^d$ open and $a, b \in \mathbb{R}^d$ the bound*

$$\sup_{E \in I, \eta \neq 0} \mathbb{E} [\|\chi_a R_{E+i\eta}(H_G) \chi_b\|^{s_0}] \leq C_{s_0} e^{-\mu_{s_0}|a-b|} \quad (A.2)$$

holds, then for every $0 < s < 1$ there exist constants $C_s, \mu_s > 0$ such that for all $G \subseteq \mathbb{R}^d$ open and $a, b \in \mathbb{R}^d$ the bound

$$\sup_{E \in I, \eta \neq 0} \mathbb{E} [\|\chi_a R_{E+i\eta}(H_G) \chi_b\|^s] \leq C_s e^{-\mu_s|a-b|} \quad (A.3)$$

holds.

PROOF. For every $0 < s \leq s_0$, the claim (A.3) follows from (A.2) by Hölder's inequality. For $s_0 < s < 1$ we have $\frac{1-s_0}{s-s_0} > 1$. We fix $q \in (1, \frac{1-s_0}{s-s_0})$ and let $p > 1$ be its Hölder conjugate. Using Hölder's inequality, we estimate for all $z \in \mathbb{C} \setminus \mathbb{R}$, $a, b \in \mathbb{R}^d$ and all $G \subseteq \mathbb{R}^d$ open

$$\begin{aligned} \mathbb{E} [\|\chi_a R_z(H_G) \chi_b\|^s] &= \mathbb{E} \left[\|\chi_a R_z(H_G) \chi_b\|^{s_0/p} \|\chi_a R_z(H_G) \chi_b\|^{s-s_0/p} \right] \\ &\leq \mathbb{E} [\|\chi_a R_z(H_G) \chi_b\|^{s_0}]^{1/p} \mathbb{E} [\|\chi_a R_z(H_G) \chi_b\|^{qs-s_0q/p}]^{1/q}. \end{aligned} \quad (\text{A.4})$$

The choice of q ensures that $qs - s_0q/p = q(s - s_0) + s_0 < 1$, and we use the a priori estimate (2.7) to bound the second expectation in the last line of (A.4). Finally, assumption (A.2) yields the claim. \square

Remark A.3. The proof of Lemma A.2 shows that if one replaces H_G by H_G^τ and assumes that (A.2) holds even uniformly in $\tau \in [0, 1]$, then the assertion also holds uniformly in $\tau \in [0, 1]$.

PROOF OF LEMMA A.1. Let $z := E + i\eta$ with $E \in I$ and $\eta \neq 0$. For the moment, we also fix $0 < s < 1/2$. The resolvent equation

$$R_z(H_G^\tau) = R_z(H_G) - \tau R_z(H_G) W R_z(H_G^\tau) \quad (\text{A.5})$$

yields the upper bound

$$\begin{aligned} \mathbb{E} [\|\chi_a R_z(H_G^\tau) \chi_b\|^s] &\leq \mathbb{E} [\|\chi_a R_z(H_G) \chi_b\|^s] + \tau^s \mathbb{E} [\|\chi_a R_z(H_G) W R_z(H_G^\tau) \chi_b\|^s] \\ &=: I_1 + \tau^s I_2. \end{aligned} \quad (\text{A.6})$$

While the first term I_1 is independent of τ and can directly be estimated by the fractional-moment bound (2.6) for the unperturbed operator H_G , we estimate I_2 as

$$\begin{aligned} I_2 &\leq \|W\|_\infty^s \sum_{c \in \Gamma_W} \mathbb{E} [\|\chi_a R_z(H_G) \chi_c\|^s \|\chi_c R_z(H_G^\tau) \chi_b\|^s] \\ &\leq \|W\|_\infty^s \sum_{c \in \Gamma_W} \mathbb{E} [\|\chi_a R_z(H_G) \chi_c\|^{2s}]^{1/2} \mathbb{E} [\|\chi_c R_z(H_G^\tau) \chi_b\|^{2s}]^{1/2}, \end{aligned} \quad (\text{A.7})$$

where Γ_W is defined in (4.20). Now, we estimate the first expectation with the fractional-moment bound from Lemma A.2 for the unperturbed operator H_G and the exponent $2s < 1$. And we estimate the second expectation with the a priori estimate (2.7) which is uniform in $\tau \in [0, 1]$. This yields

$$I_2 \leq C_1 \sum_{c \in \Gamma_W} e^{-\mu_1 |c-a|} \leq C_2 e^{-\mu |a|} \quad (\text{A.8})$$

with finite constants $C_1, C_2 > 0$ that are independent of $G \subseteq \mathbb{R}^d$ open, $a \in \mathbb{R}^d$ and $\tau \in [0, 1]$. For the second inequality we used that Γ_W is finite due to the compact support of W . Since $R_z(H_G) W R_z(H_G^\tau) = R_z(H_G^\tau) W R_z(H_G)$, we obtain along the same lines that $I_2 \leq C_2 e^{-\mu |b|}$. Multiplying this inequality with (A.8), we infer

$$I_2 \leq C_2 e^{-(\mu/2)(|a|+|b|)} \leq C_2 e^{-(\mu/2)(|a-b|)}. \quad (\text{A.9})$$

Together with the bound for I_1 , which decays exponentially in $|a-b|$, we obtain the assertion for $s < 1/2$. For general $0 < s < 1$, the result then follows from Lemma A.2 together with Remark A.3. \square

We finish this appendix with another simple stability result. Eigenvalues of ergodic random Schrödinger operators move wildly as a function of the random coupling constants so that the probability for a given energy to be an eigenvalue is zero. This property does not rely on ergodicity, it follows already from the existence of the a priori bounds (2.7) and thus extends to more general situations.

Lemma A.4. *Let $E \in \mathbb{R}$ and $\tau \in [0, 1]$ be given. Then E is not an eigenvalue of H^τ almost surely.*

PROOF. Since, for $x \in \mathbb{R}$, $-i\varepsilon(x - E - i\varepsilon)^{-1} \rightarrow \mathbb{1}_{\{E\}}(x)$ as $\varepsilon \rightarrow 0$, the spectral theorem implies the strong convergence $-i\varepsilon R_{E+i\varepsilon}(H^\tau) \rightarrow \mathbb{1}_{\{E\}}(H^\tau)$ as $\varepsilon \rightarrow 0$. Using this, we obtain for some $0 < s < 1$ and any $a, b \in \mathbb{Z}^d$ that

$$\mathbb{E}[\|\chi_a \mathbb{1}_{\{E\}}(H^\tau) \chi_b\|^s] \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^s \mathbb{E}[\|\chi_a R_{E+i\varepsilon}(H^\tau) \chi_b\|^s] = 0, \quad (\text{A.10})$$

where the last equality follows from the a priori bound (2.7). Since \mathbb{Z}^d is countable, $\mathbb{1}_{\{E\}}(H^\tau) = 0$ almost surely. \square

Appendix B. Deterministic a priori estimate for Schrödinger operators

This appendix contains a local Schatten-von Neumann-class bound for functions of (non-random) Schrödinger operators. We state it in Lemma B.2 and apply it in the proofs of Theorem 3.1 and Theorem 3.2. Estimates of this type appeared in [Sim82, Sect. B.9], [AENSS06, App. A] and [BNSS06]. We include the proofs for completeness, following closely [BNSS06].

We frequently use the following inequalities for Schatten norms.

Lemma B.1. *Let $p > 0$. Then, for $A \in \mathcal{S}^\infty$ and $B \in \mathcal{S}^p$ we have*

$$\|AB\|_p^p \leq \|A\|^p \|B\|_p^p. \quad (\text{B.1})$$

If $0 < \varepsilon < p$ is such that $A \in \mathcal{S}^{p-\varepsilon}$, then the inequality

$$\|A\|_p^p \leq \|A\|^\varepsilon \|A\|_{p-\varepsilon}^{p-\varepsilon} \quad (\text{B.2})$$

holds.

PROOF. An application of the min-max principle implies

$$\mu_n(AB) \leq \|A\| \mu_n(B) \quad (\text{B.3})$$

for the singular values μ_n , $n \in \mathbb{N}$, see e.g. [Sim05b, Thm. 1.6]. Hence,

$$\|AB\|_p^p = \sum_{n \in \mathbb{N}} |\mu_n(AB)|^p \leq \|A\|^p \|B\|_p^p \quad (\text{B.4})$$

holds. Moreover, we have

$$\|A\|_p^p = \sum_{n \in \mathbb{N}} \mu_n(A)^p = \sum_{n \in \mathbb{N}} \mu_n(|A|)^p = \sum_{n \in \mathbb{N}} \mu_n(|A|^{\varepsilon/p} |A|^{1-\varepsilon/p})^p. \quad (\text{B.5})$$

Together with (B.3), this gives (B.2). \square

In the remaining part of this appendix we consider the non-random Schrödinger operator

$$(D) \quad H := -\Delta + U \text{ with a bounded potential } U \in L^\infty(\mathbb{R}^d),$$

acting in $L^2(\mathbb{R}^d)$. To formulate the deterministic a priori bound we introduce $E_0 := \inf_{x \in \mathbb{R}^d} U(x)$ so that $H_G \geq E_0$ for all $G \subseteq \mathbb{R}^d$ open.

Lemma B.2. *Assume (D). Let $p > 0$ and $m \in \mathbb{N}$ with $m > d/(2p)$. Then there exists a finite constant C such that for all open $G \subseteq \mathbb{R}^d$, all $a, b \in \mathbb{R}^d$ and all $g \in L^\infty(\mathbb{R})$ we have the estimate*

$$\|\chi_a g(H_G) \chi_b\|_p \leq C \|g\|_\infty (\max\{0, \sup \text{supp}(g) - E_0 + 1\})^m. \quad (\text{B.6})$$

The constant C in (B.6) depends on the potential U , but only through $\inf \sigma(H) - E_0$. The proof of Lemma B.2 is based, up to some iteration procedure, on the following Schatten-class Combes-Thomas estimate.

Lemma B.3. *Assume (D). Let $p > d/2$, $E \in (-\infty, E_0)$ and $\tilde{E} := E_0 - E > 0$. Then there exist finite constants $C_{p, \tilde{E}}, \mu_{p, \tilde{E}} > 0$ such that for all open $G \subseteq \mathbb{R}^d$ and all $a, b \in \mathbb{R}^d$*

$$\|\chi_a (H_G - E)^{-1} \chi_b\|_p \leq C_{p, \tilde{E}} e^{-\mu_{p, \tilde{E}} |a-b|}. \quad (\text{B.7})$$

Whereas the constant $\mu_{p, \tilde{E}}$ in (B.7) is independent of the potential U , the constant $C_{p, \tilde{E}}$ depends on U , but only through $\inf \sigma(H) - E_0$.

PROOF OF LEMMA B.3. Let $E \in (-\infty, E_0)$ and $G \subseteq \mathbb{R}^d$ open. The form inequality

$$H_G \geq -\Delta + E_0 \quad (\text{B.8})$$

and Eq. (2.21) in [Kat95, Sec. VI.2] imply the bound

$$\|(-\Delta + E_0 - E)^{1/2} (H_G - E)^{-1/2}\| \leq 1 \quad (\text{B.9})$$

for $E < E_0$. We set $\tilde{E} := E_0 - E$. Using Hölder's inequality, we estimate for fixed $a \in \mathbb{R}^d$ and $p' \geq 1$ (to be determined later)

$$\begin{aligned} \|\chi_a (H_G - E)^{-1/2}\|_{p'} &= \|\chi_a (-\Delta + \tilde{E})^{-1/2} (-\Delta + \tilde{E})^{1/2} (H_G - E)^{-1/2}\|_{p'} \\ &\leq \|\chi_a (-\Delta + \tilde{E})^{-1/2}\|_{p'} =: C_{p', \tilde{E}}, \end{aligned} \quad (\text{B.10})$$

where $C_{p', \tilde{E}} < \infty$ is finite for $p' > d$ [Sim82, Thm. B.9.3] and independent of a . Now let $p > d/2$ and fix $\theta \in (0, p - d/2)$, whence $p' := 2(p - \theta) > d$. We obtain with Lemma B.1

$$\|\chi_a (H_G - E)^{-1} \chi_b\|_p^p \leq \|\chi_a (H_G - E)^{-1} \chi_b\|_{p-\theta}^{p-\theta} \|\chi_a (H_G - E)^{-1} \chi_b\|^\theta. \quad (\text{B.11})$$

Hölder's inequality and the bound (B.10) therefore imply

$$\begin{aligned} \|\chi_a (H_G - E)^{-1} \chi_b\|_p^p &\leq \|\chi_a (H_G - E)^{-1/2}\|_{2(p-\theta)}^{p-\theta} \|(H_G - E)^{-1/2} \chi_b\|_{2(p-\theta)}^{p-\theta} \\ &\quad \times \|\chi_a (H_G - E)^{-1} \chi_b\|^\theta \\ &\leq C_{2(p-\theta), \tilde{E}}^{2(p-\theta)} \|\chi_a (H_G - E)^{-1} \chi_b\|^\theta. \end{aligned} \quad (\text{B.12})$$

Now, the finite-volume operator can be replaced by the infinite-volume operator

$$\|\chi_a (H_G - E)^{-1} \chi_b\| \leq \|\chi_a (H - E)^{-1} \chi_b\|. \quad (\text{B.13})$$

This follows from the Feynman-Kac representation for Dirichlet restrictions, see e.g. [BHL00, Sect. 6], which implies the estimate

$$0 \leq (\chi_a (H_G - E)^{-1} \chi_b)(x, y) \leq (\chi_a (H - E)^{-1} \chi_b)(x, y) \quad (\text{B.14})$$

for the kernels and all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Moreover, we use $\|A\psi\| \leq \|B|\psi|\| \leq \|B\|\|\psi\|$ for integral operators A and B with kernels $0 \leq A(x, y) \leq B(x, y)$, proving (B.13). Finally, the Combes-Thomas estimate [GK03, Eq. (19)] gives

$$\|\chi_a(H - E)^{-1}\chi_b\| \leq C_{CT} e^{-\mu_{CT}|a-b|} \quad (\text{B.15})$$

with finite constants $\mu_{CT} > 0$, which depends only on \tilde{E} , and C_{CT} , which depends only on \tilde{E} and on $\inf \sigma(H) - E_0$. The claim follows from (B.11), (B.12), (B.13) and (B.15). \square

PROOF OF LEMMA B.2. We use the abbreviation $H := H_G$. Without loss of generality we assume $E_0 \leq \sup \text{supp}(g) < \infty$ (because otherwise the statement is trivial) and $0 < p \leq 1$ (because $\|\cdot\|_p \leq \|\cdot\|_1$ for $p \geq 1$). Let $m \in \mathbb{N}$ such that $m > d/(2p)$. We insert the m -th power of the resolvent and apply Hölder's inequality

$$\begin{aligned} \|\chi_a g(H) \chi_b\|_p^p &= \|\chi_a (H - E_0 + 1)^{-m} (H - E_0 + 1)^m g(H) \chi_b\|_p^p \\ &\leq \|\chi_a (H - E_0 + 1)^{-m}\|_p^p \|(H - E_0 + 1)^m g(H) \chi_b\|_p^p \\ &\leq (\sup \text{supp}(g) - E_0 + 1)^{mp} \|g\|_\infty^p \|\chi_a (H - E_0 + 1)^{-m}\|_p^p. \end{aligned} \quad (\text{B.16})$$

Next we set $a_0 := a$ and estimate with the adapted triangle inequality (3.2)

$$\begin{aligned} \|\chi_a (H - E_0 + 1)^{-m}\|_p^p &= \left\| \chi_{a_0} (H - E_0 + 1)^{-1} \left(\sum_{a_1 \in \mathbb{Z}^d} \chi_{a_1} \right) (H - E_0 + 1)^{-1} \right. \\ &\quad \times \cdots \left(\sum_{a_{m-1} \in \mathbb{Z}^d} \chi_{a_{m-1}} \right) (H - E_0 + 1)^{-1} \left(\sum_{a_m \in \mathbb{Z}^d} \chi_{a_m} \right) \Big\|_p^p \\ &\leq \sum_{a_1, \dots, a_m \in \mathbb{Z}^d} \left\| \prod_{l=1}^m (\chi_{a_{l-1}} (H - E_0 + 1)^{-1} \chi_{a_l}) \right\|_p^p. \end{aligned} \quad (\text{B.17})$$

Hölder's inequality for Schatten classes and the Combes-Thomas estimate in the form of Lemma B.3 with $pm > d/2$ and $E = E_0 - 1$ imply

$$\begin{aligned} \left\| \prod_{l=1}^m (\chi_{a_{l-1}} (H - E_0 + 1)^{-1} \chi_{a_l}) \right\|_p^p &\leq \prod_{l=1}^m \|\chi_{a_{l-1}} (H - E_0 + 1)^{-1} \chi_{a_l}\|_{pm}^p \\ &\leq C_1 \prod_{l=1}^m e^{-\mu_1 |a_{l-1} - a_l|} \end{aligned} \quad (\text{B.18})$$

with finite constants $\mu_1 = \mu_{1,p,m} > 0$ and $C_1 = C_{1,p,m} > 0$. The latter also depends on the potential U , but only through $\inf \sigma(H) - E_0$. Inserting (B.18) into (B.17) and repeatedly using

$$\sum_{a_1 \in \mathbb{Z}^d} e^{-\mu_1 |a_0 - a_1|} e^{-\mu_1 |a_1 - a_2|} \leq C_2 e^{-\mu_1/2 |a_0 - a_2|}, \quad (\text{B.19})$$

where $C_2 = C_{2,p,m} > 0$ is a finite constant, we obtain the lemma from (B.16). \square

Appendix C. The index of a pair of projections

The results in this appendix are essentially contained in [ASS94]. We collect them here together with their short proofs for the convenience of the reader. They will mainly be needed in the proof of Theorem 3.19.

We start with a representation for powers of the difference of orthogonal projections.

Lemma C.1. *Let P and Q be two orthogonal projections and let $P^c := \mathbb{1} - P$, $Q^c := \mathbb{1} - Q$. Then, the formulas*

$$(P - Q)^{2n-1} = (PQ^c)^n - (P^cQ)^n \quad (\text{C.1})$$

and

$$(P - Q)^{2n} = (PQ^cP)^n + (P^cQP^c)^n \quad (\text{C.2})$$

hold for each $n \in \mathbb{N}$.

PROOF. We compute

$$P - Q = P(Q^c + Q) - (P + P^c)Q = PQ^c - P^cQ \quad (\text{C.3})$$

and

$$(P - Q)^2 = (P - Q)(P - Q)^* = PQ^cP + P^cQP^c. \quad (\text{C.4})$$

Formula (C.2) follows from iterated multiplications of (C.4) with itself. Formula (C.1) follows from multiplying the $(n - 1)$ st power of (C.4) with (C.3). \square

For the next lemma we recall the definition of the spectral shift operator $T(E, A, B)$ from (3.12) and of the index $\theta(E, A, B)$ from (3.15).

Lemma C.2. *Let A, B be self-adjoint operators in a Hilbert space, which are both bounded from below. Assume that the perturbation $B - A$ is sign-definite, that is, there exists $\alpha \in \{\pm 1\}$ such that $\alpha(B - A) \geq 0$. Assume further that the spectral shift operator obeys $T(E, A, B) \in \mathcal{S}^1$ for some $E \in \mathbb{R}$. Then, we have*

$$\dim \ker (T(E, A, B) + \alpha \mathbb{1}) = 0 \quad (\text{C.5})$$

and

$$\theta(E, A, B) = \alpha \dim \ker (T(E, A, B) - \alpha \mathbb{1}). \quad (\text{C.6})$$

In particular, we have

$$1 \in \sigma(T(E, A, B)^2) \iff \theta(E, A, B) \neq 0. \quad (\text{C.7})$$

PROOF OF LEMMA C.2. The implication “ \Leftarrow ” in (C.7) follows immediately from (C.6); for “ \Rightarrow ”, use (C.6) and (C.5). The equality (C.6) follows from the definition (3.15) of the index and (C.5). Thus, it remains to prove (C.5). We restrict ourselves to the case $B - A \geq 0$ and define the linear subspace

$$\mathcal{V}_- := \ker (\mathbb{1}_{(-\infty, E]}(A)) \cap \text{ran} (\mathbb{1}_{(-\infty, E]}(B)). \quad (\text{C.8})$$

Clearly, $\mathcal{V}_- \subseteq \ker (T(E, A, B) + \mathbb{1})$. Conversely, let $\psi \in \ker (T(E, A, B) + \mathbb{1})$, then

$$\langle \psi, \mathbb{1}_{(-\infty, E]}(A)\psi \rangle + \langle \psi, (1 - \mathbb{1}_{(-\infty, E]}(B))\psi \rangle = 0, \quad (\text{C.9})$$

which implies $\psi \in \mathcal{V}_-$. This proves $\mathcal{V}_- = \ker (T(E, A, B) + \mathbb{1})$. It remains to show that $\mathcal{V}_- = \{0\}$. Pick $E_0 \in \mathbb{R}$ such that $A, B > E_0$. If $E \leq E_0$ then $\text{ran} (\mathbb{1}_{(-\infty, E]}(B)) = \{0\}$ and so $\mathcal{V}_- = \{0\}$. Now, consider the case $E > E_0$. We assume that there exists $\psi \in \mathcal{V}_-$ with $\|\psi\| = 1$. Thus we have both $\psi \in \ker (\mathbb{1}_{(-\infty, E]}(A)) = \text{ran} (\mathbb{1}_{(E, \infty)}(A))$ and $\psi \in \text{ran} (\mathbb{1}_{(-\infty, E]}(B))$. Using operator-monotonicity of the resolvent below the spectrum, we conclude

$$(E - E_0)^{-1} \leq \langle \psi, R_{E_0}(B)\psi \rangle \leq \langle \psi, R_{E_0}(A)\psi \rangle < (E - E_0)^{-1}, \quad (\text{C.10})$$

which is a contradiction. \square

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